

ASYMPTOTIC TOPOLOGY

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ABSTRACT. We establish some basic theorems in dimension theory and absolute extensor theory in the coarse category of metric spaces. Some of the statements in this category can be translated in general topology language by applying the Higson corona functor. The relation of problems and results of this ‘Asymptotic Topology’ to Novikov and similar conjectures is discussed.

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§1 INTRODUCTION

The large scale geometry became very popular mainly thanks to Gromov. It is a basic tool of study of discrete groups and geometric and topological invariants of manifolds associated with the fundamental group [Gr1],[Ro1], [Ro2]. Objects of study in the large scale geometry are unbounded metric spaces, such as complete open Riemannian manifolds, their global pictures, where small (bounded) details are not taken into account. One investigates the properties there which are defined by taking the limit at ‘infinity’. This makes an analogy between the large scale world and the small scale world, where one takes the limit at 0. A significant piece of Topology is devoted to study local properties of spaces. i.e. the small scale world. The most developed are the theory of local connectivity, ANR-theory, dimension theory and cohomological dimension of compact metric spaces. The purpose of this paper is an attempt to transfer these theories to the large scale world. Partly it was already done by other authors. Thus the basic ideas of large scale dimension theory were introduced by Gromov. The coarse (large scale) cohomology groups were defined by Roe. An importance of large scale versions of concepts of local topology was known for years. The most striking result is due to G.Yu [Y1] and it says that the Novikov higher signature conjecture holds for geometrically finite groups with a finite large scale dimension. We note that a discrete group Γ has a natural metric whenever one fixes a set of generators on it. For finitely generated groups any two such metrics defined by means of finite sets of generators are equivalent from the large scale point of view.

Many properties of discrete groups or universal covers of their classifying spaces can be detected from the boundaries of the groups. So far there is no good construction of a boundary of arbitrary group. We refer to [Be] for the axioms and some constructions. The visual sphere at infinity is good only in hyperbolic case. In a semi-hyperbolic case it is not a coarse invariant [B-R],[C-K]. Nevertheless there is the boundary of a group which is coarse invariant, namely, the Higson corona. It can be defined for any proper metric space X , i.e. a metric space where the distance to a point is a proper function, and it is a covariant functor ν to the category of compact spaces in corresponding setting. The problem with the Higson corona is that it is never metrizable. Thus in order to study coarse (asymptotic) topology of nice spaces one forced to deal with invariants of the topology of nonmetrizable compact Hausdorff spaces. Many of the technical tools of this branch of general topology such as Mardešić factorization theorem or Shchepin’s spectral theorem are important here. The Higson corona makes a functorial connection between the large scale world and an exotic small scale world which is not completely satisfactory. One of the problems is that the cohomology groups of Higson corona is not a homotopy invariant of coarse topology. In the coarse topology the spaces \mathbf{R}^n and the n -dimensional hyperbolic space are homotopy equivalent whereas their Higson coronas have different n -cohomologies [Dr-F].

The main motivation for studying the macroscopic topology is the Novikov higher signature conjecture. The conjecture claims that the higher signatures of a manifold are homotopy invariant. For aspherical manifolds it can be rephrased that the rational Pontryagin classes are homotopy invariant. Novikov's theorem says that the rational Pontryagin classes are topological invariant [N]. Generally rational Pontryagin classes are not homotopy invariant. The term higher signature makes its origin from the Hirzebruch signature formula: $\langle L_k(p_1, \dots, p_k), [M^{4k}] \rangle = \sigma(M^{4k})$ where $L_k(p_1, \dots, p_k)$ is the k -th Hirzebruch polynomial in Pontryagin classes and $[M^{4k}]$ is the fundamental class. For a simply connected manifold M the signature $\sigma(M)$ is the only homotopy invariant which can be written in terms of Pontryagin classes. In the presence of the fundamental group $\Gamma = \pi_1(M)$ there are additional possibilities. In that case besides the fundamental class $[M]$ there are other 'special' homology classes which comes from the homology of the group Γ . Integration of the Hirzebruch polynomials on these classes defines the *higher signatures* of a manifold.

The restriction to the case of aspherical manifolds M is not that strong as it seems. It covers the Novikov conjecture for manifolds with all geometrically finite fundamental groups. We recall that a group Γ is *geometrically finite* if $B\Gamma = K(\Gamma, 1)$ is a finite complex. In that case the universal cover $E\Gamma$ is contractible and coarsely equivalent to Γ .

Another serious conjecture about aspherical manifolds is the Gromov-Lawson conjecture [G-L]: *An aspherical manifold cannot carry a metric of a positive scalar curvature.* The connection between this and the Novikov conjecture is discussed in [Ros], [F-R-R].

Let X be the universal cover of an aspherical manifold M^n supplied with a metric lifted from M^n . Then the above venerable conjectures can be reduced to some large scale problems about the metric space X . As a topological space X is a contractible n -manifold. Hence $X \times \mathbf{R}$ is homeomorphic to \mathbf{R}^{n+1} . Without loss of generality we may assume in the above problems that $X = \mathbf{R}^n$. All problems are concentrated in the metric on X . In §8 we consider several conjectures about X , the validity of which would imply the Novikov and the Gromov-Lawson conjectures.

It turns out that several statements of general topological nature formulated in macroscopic world imply the Novikov (Gromov-Lawson) conjecture. We discuss here mainly dimensional theoretic statements. As it was mentioned, Yu proved [Y1] that if the macroscopic dimension of a geometrically finite group with a word metric on it is finite then the Novikov and the Gromov-Lawson conjecture holds for manifolds with the fundamental group Γ . In this paper we define the notion of macroscopic cohomological dimension $as\dim_{\mathbf{Z}} \Gamma$ and show that $as\dim_{\mathbf{Z}} \Gamma < \infty$ for all geometrically finite groups. This reduces the Novikov conjecture to the old Alexandroff problem about an equivalence of two concepts of dimension: geometrical (Lebesgue's dimension) and algebraic (cohomological dimension) considered in the macroscopic world. The Alexandroff Problem in classical dimension theory was resolved by a counterexample [Dr]. This counterexample was also transferred to the macroscopic world in [D-F-W] where we defined a uniformly contractible Riemannian metric on \mathbf{R}^8 in such a way that the asymptotic dimension is infinite and the macroscopic cohomological dimension is finite. Nevertheless in the large scale topology

the problem is still open in the case when a metric space is a group with word metric on it. With respect to that it would be useful to investigate the situation in classical case. For compact topological groups the equivalence of these two dimensions easy follows from their presentation as inverse limits of Lie groups. It is unclear whether compact groups in the microscopic world are proper analogs of discrete groups in the macroscopic world. It seems that the Alexandroff Problem for compact H-spaces is more relevant to the large scale world.

The body of the paper consists of §2-9. In §10 we present a list of open problems.

In §2 we define a category suitable for the macroscopic topology. We called it the Asymptotic category. Note that our category differs slightly from Roe's Coarse category. This difference is stipulated by our desire to build an analog of the ANR theory in the macroscopic world.

In §3 we define some basic constructions in the asymptotic category.

In §4 the notion of AE (absolute extensor) is defined and simple examples of AE objects are presented.

In §5 we define ANE spaces and a notion of homotopy between two morphisms. Also we discuss different approaches to definition of dimension in the asymptotic category. In particular we define a macroscopic dimension in terms of extension of maps to objects analogous to spheres.

In §6 we define the coarse cohomology by means of anti-Čech approximations by polyhedra and then we define an asymptotic cohomological dimension.

In §7 we consider the Higson corona νX and compare various forms of the asymptotic dimension of a metric space X with the covering dimension of the Higson corona νX . The main result here is that the categorical definition of asymptotic dimension of X agrees with the covering dimension of νX .

In §8 we prove some versions of coarse Novikov conjectures under some dimensional conditions on groups. Our results here are extensions of Yu's [Y1],[Y2]. In particular we prove the Coarse Baum-Connes conjecture for groups with the slow dimension growth.

In §9 we compare different coarse reduction of the rational Novikov conjecture in the language of the Higson corona.

§2 CHOICE OF CATEGORY

A metric space (X, d) is called *proper* if every closed ball $B_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$ is compact. A map $f : X \rightarrow Y$ is called *proper* if $f^{-1}(C)$ is compact for every compact set $C \subset Y$. Note that a proper map between proper metric spaces is always continuous.

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ is called asymptotically Lipschitz if there are numbers λ and s such that $d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + s$.

An *asymptotic category* \mathcal{A} consists of proper metric spaces and proper asymptotically Lipschitz maps.

We consider the following refinement $\tilde{\mathcal{A}}$ of this category: Let $x_0 \in X$ be a base point, then the norm $\|x\|$ of $x \in X$ is the distance $d_X(x, x_0)$. Then the norm $\|f\|$ of a map $f : X \rightarrow Y$ is $\lim_{x \rightarrow \infty} \frac{\|f(x)\|}{\|x\|}$. It is clear that the norm $\|f\|$ does not depend on choice

of base points. A map with a nonzero norm is always proper. In the category $\tilde{\mathcal{A}}$ we consider morphisms of \mathcal{A} with nonzero norms. For any such map $f : X \rightarrow Y$ there are constants c and b such that $\|f(x)\| \geq c\|x\| - b$.

EXAMPLE 1. The Euclidean n -space \mathbf{R}^n , the half n -space \mathbf{R}_+^n with the standard metric are examples of objects in \mathcal{A} .

Following Gromov we are going to call Lipschitz maps with the Lipschitz constant one as *short maps*. A distance to a given point is a typical morphism in $\tilde{\mathcal{A}}$, $d(-, x_0) : X \rightarrow \mathbf{R}_+$ on a proper metric space (X, d) to \mathbf{R}_+ . It is a short map.

Proposition 2.1. *If $f : X \rightarrow Y$ is asymptotically Lipschitz with the Lipschitz constant λ , then for any $\alpha > 1$ there is $R > 0$ such that $d_Y(f(x), f(x')) < \alpha \lambda d_X(x, x')$ provided $d(x, x') \geq R$.*

Isomorphisms $f : X \rightarrow Y$ in the category \mathcal{A} are homeomorphisms with f and f^{-1} asymptotically Lipschitz. Isomorphisms in $\tilde{\mathcal{A}}$ are isomorphisms in \mathcal{A} with nonzero norm.

EXAMPLE 2. All Banach n -dimensional spaces are isomorphic to \mathbf{R}^n in $\tilde{\mathcal{A}}$ (and hence in \mathcal{A}).

There is a bigger category $\bar{\mathcal{A}}$ which is also of great importance. The objects of category $\bar{\mathcal{A}}$ are the same as in \mathcal{A} (actually, one can take all metric spaces) and morphisms are asymptotically Lipschitz and *coarsely proper* maps. The last means that $f^{-1}(C)$ is bounded for every bounded set $C \subset Y$. In this case f is not necessarily continuous. One can define a refinement of $\bar{\mathcal{A}}$. Two morphisms in $\bar{\mathcal{A}}$ are called *coarse equivalent* if they are within a finite distance i.e. there is $c > 0$ such that $d_Y(f(x), g(x)) < c$ for all $x \in X$. A morphism $f : X \rightarrow Y$ is called a *coarse isomorphism* if there is a morphism $g : Y \rightarrow X$ s.t. $f \circ g$ and $g \circ f$ are equivalent to 1_X and 1_Y respectively. Metric spaces X and Y are coarse isomorphic (quasi-isometric) if there is a coarse isomorphism $f : X \rightarrow Y$. The quotient category $\mathcal{C} = \bar{\mathcal{A}} / \sim$ is called the *coarse category*.

We recall that the Gromov-Hausdorff distance between metric spaces $d_{GH}(X, Y)$ is the infimum of distances between subsets $i(X), j(Y) \subset Z$ for all possible isometric imbeddings i, j for all possible metric spaces Z . Very often $d_{GH}(X, Y) = \infty$.

Proposition 2.2. *If $d_{GH}(X, Y) < \infty$ then X and Y are coarsely isomorphic.*

Note that a map which is only a finite distance apart from asymptotically Lipschitz map is asymptotically Lipschitz itself.

DEFINITION. A metric space X is called *uniformly contractible*, we denote $X \in UC$, if there is a function $S : [0, \infty) \rightarrow \mathbf{R}$ such that every ball $B_r(x)$ of radius r centered at x can be contracted to a point in the ball $B_{S(r)}(x)$.

Proposition 2.3. *Let Z, X be objects of \mathcal{A} and let $\dim Z < \infty$ and $X \in UC$. Then every morphism $f : Z \rightarrow X$ in $\bar{\mathcal{A}}$ is coarsely isomorphic to a morphism in \mathcal{A} .*

Proof. Let $\dim Z = n$, let λ and s be the constants of f from the definition of asymptotically Lipschitz condition and let S be a function from the definition of UC property of X . Let \mathcal{U} be an open covering of Z of order $\leq n + 1$ by sets of diameter ≤ 1 . Let $\phi : Z \rightarrow N$ be a projection to the nerve of the covering \mathcal{U} . We define a map $g_0 : N^{(0)} \rightarrow X$ on

0-dimensional skeleton of N by the rule: $g_0(u) = f(x_u)$ for some $x_u \in \phi^{-1}(u) \subset U$. The UC property allows to extend g_0 to a map $g : N \rightarrow X$ in such way that the image of a k -dimensional simplex $g(\sigma^k)$ is contained in the S_k -neighborhood of $g_0((\sigma^k)^{(0)})$, where $S_k = S(2S(\dots S(\lambda + s))$ is a number obtained after k iterations. Then for given $z \in Z$ we have $d(f(z), g\phi(z)) \leq d(f(z), g_0(u)) + d(g_0(u), g\phi(z))$. We take u such that $\phi(z)$ and u are in the same simplex. Then $d(f(z), g\phi(z)) \leq d(f(z), f(x_u)) + S_n \leq 1 + S_n = c$. \square

REMARK. In the above proof instead of $\dim Z < \infty$ we can use a weaker condition: Z admits a cover of finite order by uniformly bounded sets.

In [Roe2] the coarse category is defined with slightly different morphisms. A map $f : X \rightarrow Y$ is a coarse map if it is coarse proper and coarsely uniform i.e. there is a positive function $S : [0, \infty) \rightarrow \mathbf{R}$ such that

$$d_X(x, x') \leq r \Rightarrow d_Y(f(x), f(x')) \leq S(r)d_X(x, x').$$

It is clear that this property is weaker than asymptotically Lipschitz condition.

We recall that a metric space (X, d) is called a *geodesic metric space* if for every two points $x, y \in X$ there is an isometric imbedding $j : [0, d(x, y)] \rightarrow X$ with $j(0) = x$ and $j(d(x, y)) = y$.

Proposition 2.4. *Let X be a geodesic metric space, then every coarse map $f : X \rightarrow Y$ is asymptotically Lipschitz.*

Proof. We take $s = S(1)$ and $\lambda = S(1)$. If $d(x, x') = n + \alpha$, $0 \leq \alpha < 1$, then $d_Y(f(x), f(x')) \leq d_Y(f(x), f(j(1))) + \dots + d_Y(f(j(n-1)), f(j(n))) + d_Y(f(j(n)), f(x')) \leq S(1)n + S(1) \leq \lambda n + s$. Here $j : [0, d(x, x')] \rightarrow X$ is a geodesic segment joining x and x' .

Nevertheless there are situations when Roe's morphisms are more appropriate. For example, they give a richer embedding theory. A morphism $j : Y \rightarrow X$ is called an *imbedding* in \mathcal{A} if j is an injection, $j(Y)$ is a closed subset and $j^{-1} : (j(Y), d_X|_{j(Y)}) \rightarrow Y$ is a morphism in \mathcal{A} , i.e. asymptotically Lipschitz. Thus, a bending a line into parabola with the induced metric from the plane is not an embedding of a line into \mathbf{R}^2 in \mathcal{A} and it is an embedding in Roe's sense.

The main flaw of coarsely uniform maps is that there is no good extension theory for them (see Remark 2 in §4). We prefer asymptotically Lipschitz maps in the asymptotic category because coarsely uniform maps makes weaker the analogy between proper metric spaces and compacta. Anyway, Proposition 2.4 shows that our category and Roe's are very close.

§3 SOME FUNCTORS IN THE ASYMPTOTIC CATEGORY

1. Product. The *Cartesian product* $X \times Y$ of two spaces in \mathcal{A} is well defined. The metric on the product can be taken as follows $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ which is equivalent to the 'euclidean' metric $\sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$. The problem with this product is that it is not categorical. The projections onto the factors are not morphisms in \mathcal{A} . Here we define an *asymptotic product* $X \tilde{\times} Y$ as pull-back in the

topological category in the following diagram:

$$\begin{array}{ccc} X \tilde{\times} Y & \xrightarrow{f} & Y \\ g \downarrow & & d_Y \downarrow \\ X & \xrightarrow{d_X} & \mathbf{R}_+ \end{array}$$

Here $d_X(x) = d_X(x, x_0)$ and $x_0 \in X$ is a base point. The metric on $X \tilde{\times} Y$ is taken from the product $X \tilde{\times} Y \subset X \times Y$. This definition is good for connected spaces. Generally this operation could give an empty space.

Proposition 3.1. *Let X and Y be geodesic metric spaces and let $Z(x_0, y_0)$ denote the asymptotical product $X \tilde{\times} Y$ defined by means of the base points $x_0 \in X$ and $y_0 \in Y$. Then for any points $x'_0 \in X$ and $y'_0 \in Y$ the spaces $Z(x_0, y_0)$ and $Z(x'_0, y'_0)$ are in a finite Gromov-Hausdorff distance.*

Proof. Let $(x, y) \in Z(x_0, y_0)$. Then by the definition $d_X(x, x_0) = d_Y(y, y_0)$. Let $d = \min\{d_X(x, x_0), d_X(x, x'_0), d_Y(y, y'_0)\}$. Consider geodesics $[x'_0, x]$ in X and $[y'_0, y]$ in Y and take points $x' \in [x'_0, x]$ and $y' \in [y'_0, y]$ with $d_X(x'_0, x') = d = d_Y(y'_0, y')$. Then $(x', y') \in Z(x'_0, y'_0)$. Note that $d_{X \times Y}((x, y), (x', y')) = d_X(x, x') + d_Y(y, y') = d_X(x, x'_0) - d + d_Y(y, y'_0) - d \leq |d_X(x'_0, x) - d_X(x, x_0)| + |d_Y(y'_0, y) - d_Y(y, y_0)| \leq d_X(x_0, x'_0) + d_Y(y_0, y'_0) = R$. Therefore (x, y) lies in the R -neighborhood of $Z(x'_0, y'_0)$. Hence $Z(x_0, y_0)$ lies in the R -neighborhood of $Z(x'_0, y'_0)$. Similarly, $Z(x'_0, y'_0)$ lies in the R -neighborhood of $Z(x_0, y_0)$. \square

Thus, by Proposition 2.2 the class of $X \tilde{\times} Y$ in the coarse category does not depend on choice of base point.

As it will be seen latter, the half line \mathbf{R}_+ plays the role of point in the category \mathcal{A} . Hence the above definition is quite logical. Note that $X \tilde{\times} \mathbf{R}_+ = X$.

Since the product with unit interval turns into the identity functor after coarsening, we need a modified notion of the product with interval. Again as it will be shown latter, the half plane \mathbf{R}_+^2 is an analog of the unit interval in \mathcal{A} . Then $X \tilde{\times} \mathbf{R}_+^2$ substitutes for the product with unit interval. Let $x_0 \in X$ be a base point. We define a map $J : X \tilde{\times} \mathbf{R}_+^2 \rightarrow X \times \mathbf{R}$ by the formula: $J(x, (s, t)) = (x, s)$ where $t \in \mathbf{R}_+$ and $s \in \mathbf{R}$.

Proposition 3.2. *J is an imbedding in the category \mathcal{A} .*

Proof. First, the image imJ consists of the region in $X \times \mathbf{R}$ between the graphs of the functions d_X and $-d_X$. Show that $J : X \tilde{\times} \mathbf{R}_+^2 \rightarrow imJ$ is bi-Lipschitz. Note that $d_{X \times \mathbf{R}}(J(x, (s, t)), J(x', (s', t'))) = d_X(x, x') + |s - s'| \leq d_X(x, x') + |s - s'| + |t - t'| = d_{X \times \mathbf{R}_+^2}((x, (s, t)), (x', (s', t')))$. On the other hand, since $\|x\| = t + |s|$ and $\|x'\| = t' + |s'|$, we have

$$2d_{X \times \mathbf{R}_+^2}(J(x, (s, t)), J(x', (s', t'))) = 2d_X(x, x') + 2|s - s'| \geq d_X(x, x') + ||x\| - \|x'\|| + ||s| - |s'||| + |s - s'| \geq d_X(x, x') + |t - t'| + |s - s'| = d_{X \times \mathbf{R}_+^2}((x, (s, t)), (x', (s', t'))). \quad \square$$

We define two maps $i_{\pm} : X \rightarrow X \tilde{\times} \mathbf{R}_+^2$ by the formula $i_{\pm}(x) = J^{-1}(x, \pm\|x\|)$.

Proposition 3.3. *The maps i_{\pm} are imbeddings in the category $\tilde{\mathcal{A}}$ (and hence in \mathcal{A}).*

If spaces X and Y have base points, one can define the *wedge* $X \vee Y$ by setting $d(x, y) = d_X(x, x_0) + d_Y(y_0, y)$ for $x \in X$ and $y \in Y$.

2. Quotient space. Let $f : X \rightarrow Y$ be a proper map of a metric space. We define a metric d_Y (*quotient metric*) on Y such that f will be a morphism. First we define a function $s_Y : Y \times Y \rightarrow \mathbf{R}_+$ by the formula

$$s_Y(y, y') = d_X(f^{-1}(y), f^{-1}(y')) = \inf\{d_X(x, x') \mid x \in f^{-1}(y), x' \in f^{-1}(y')\}.$$

Since a proper map is always surjective, the function s_Y is well defined. Now we define a metric d_Y as the intrinsic metric generated by s_Y :

$$d_Y(y, y') = \inf\{\sum_{i=1}^n s_Y(y_i, y_{i+1}) \mid n \in \mathbb{N}; y_1, \dots, y_n \in Y, y_1 = y, y_n = y'\}.$$

Then $d_Y(f(x), f(x')) \leq s_Y(f(x), f(x')) \leq d_X(x, x')$ and hence f is a short map.

3. Cone. In this section we define the *cone* over a proper metric space X . We identify $X \tilde{\times} \mathbf{R}_+^2$ with its image under the imbedding J . Then on $i_+(X) = \{(x, \|x\|) \in X \times \mathbf{R}_+\}$ we consider a function f' defined by the formula $f'((x, \|x\|)) = \|x\|$. We extend this map to a proper map $f : X \tilde{\times} \mathbf{R}_+^2 \rightarrow Y$ by adding singletons to the decomposition generated by $(f')^{-1}$. Then by the definition the quotient space Y with the quotient metric d_Y is the cone CX over X .

Lemma 3.4. *The cone $C(\mathbf{R}^n)$ is isomorphic to \mathbf{R}_+^{n+1} in $\tilde{\mathcal{A}}$ (and hence in \mathcal{A}).*

Proof. Let $q : J(\mathbf{R}^n \tilde{\times} \mathbf{R}_+^2) \rightarrow C\mathbf{R}^n$ be the quotient map. We define a map $T : C(\mathbf{R}^n) \rightarrow \mathbf{R}^n \times \mathbf{R}_+$ by the formula

$$T(q(x, t)) = (\frac{\|x\| - t}{\|x\|}x, t).$$

Then the inverse transform is given by the formula:

$$T^{-1}(y, t) = q((\|y\| + t)\frac{y}{\|y\|}, t).$$

First we show that the inverse map is Lipschitz. Denote by d the metric $d_{\mathbf{R}^n \times \mathbf{R}_+}$. Then

$$\begin{aligned} d_{C(\mathbf{R}^n)}(T^{-1}(y, t), T^{-1}(y', t')) &= d_{C(\mathbf{R}^n)}(q((\|y\| + t)\frac{y}{\|y\|}, t), q((\|y'\| + t')\frac{y'}{\|y'\|}, t')) \leq \\ &\leq d((\|y\| + t)\frac{y}{\|y\|}, t), ((\|y'\| + t')\frac{y'}{\|y'\|}, t')) = \|(\|y\| + t)\frac{y}{\|y\|} - (\|y'\| + t')\frac{y'}{\|y'\|}\| + |t - t'| \\ &\text{we may assume that } \|y'\| \leq \|y\|, \text{ then we continue:} \\ &\leq (1 + \frac{t}{\|y\|})\|(y - y')\| + \|y'\|\|\frac{t}{\|y\|} - \frac{t'}{\|y'\|}\| + |t - t'| \leq \\ &2\|y - y'\| + |t - t'| + \frac{\|y'\|}{\|y\|}|t - t'| + \frac{t'}{\|y\|}\|y - y'\| \leq 3\|y - y'\| + 3|t - t'| = 3d((y, t), (y', t')). \end{aligned}$$

Let $(x, t) \in J(\mathbf{R}^n \tilde{\times} \mathbf{R}_+^2)$. Note that $d((x, t), i_+(\mathbf{R}^n)) = \|x\| - t$. It is clear that

$$d_{C(\mathbf{R}^n)}(q(x, t), q(x', t')) \geq \min\{d((x, t), (x', t')), d((x, t), i_+(\mathbf{R}^n)) + d((x't'), i_+(\mathbf{R}^n))\}.$$

We may assume that $\|x'\| \geq \|x\|$. If $t' \leq \|x\|$, then $t' - t \leq \|x\| - t$. Since always $t - t' \leq \|x'\| - t'$, we have

$$2(d((x, t), i_+(\mathbf{R}^n)) + d((x't'), i_+(\mathbf{R}^n))) = 2(\|x\| - t + \|x'\| - t') \geq$$

$$\|x\| - t + \|x'\| - t' + |t - t'| \geq d(T(q(x, t)), T(q(x', t'))).$$

On the other hand, the above argument shows that $d(T(q(x, t)), T(q(x', t'))) \leq 3\|x - x'\| + 3\|t - t'\| = 3d((x, t), (x', t'))$.

Hence, $d(T(q(x, t)), T(q(x', t'))) \leq 3d_{C(\mathbf{R}^n)}(q(x, t), q(x', t'))$, provided $t' \leq \|x\|$.

Now we assume that $t' \geq \|x\|$. It is easy to check that for any two points (z_1, t_1) and (z_2, t_2) lying in $i_+(\mathbf{R}^n)$ the sum of distances

$$d_{C(\mathbf{R}^n)}(q(x, t), q(z_1, t_1)) + d_{C(\mathbf{R}^n)}(q(z_2, t_2), q(z_1, t_1)) + d_{C(\mathbf{R}^n)}(q(x', t'), q(z_2, t_2))$$

greater or equal than $\|x'\| - \|x\| + |t' - t|$.

Therefore, $d_{C(\mathbf{R}^n)}(q(x, t), q(x', t')) \geq \|x'\| - \|x\| + |t' - t| = \|x'\| - \|x\| + t' - t$.

Since $2(t' - t) + \|x'\| - \|x\| \geq \|x\| - t + \|x'\| - t' + t' - t$, it follows that

$$2d_{C(\mathbf{R}^n)}(q(x, t), q(x', t')) \geq d(Tq(x, t), Tq(x', t')). \quad \square$$

One can similarly define the suspension ΣX .

4. Probability measures. First we recall that probability measures form a functor $P : \mathcal{A} \rightarrow \mathcal{A}$. The set of all probability measures with compact supports $P(X)$ on a metric space (X, ρ) can be given a metric space structure by means of the Kantorovich-Rubinshtein metric $\bar{\rho}(\mu_1, \mu_2) = \sup\{|\int f d\mu_1 - \int f d\mu_2| \mid f \in S(X)\}$ where $S(X)$ is the set of all short real valued functions, i.e. Lipschitz functions with Lipschitz constant one. Note that X is isometrically imbedded in $P(X)$ by means of Dirac measures δ_x . For every n there is a subfunctor $P_n : \mathcal{A} \rightarrow \mathcal{A}$ of probability measures supported at most by n points. Let (X, d) be a proper metric space with a base point x_0 . The half line space \mathbf{R}_+ has the natural base point $\{0\}$. One can define the *join* product $X * Y$ of two spaces with base points X and Y as a subfunctor of $P_2(X \vee Y)$. The natural question here whether one can interpret the cone CX as the join of X and \mathbf{R}_+ .

For a proper metric space X we denote by $L(X)$ the space of all short continuous functions on X with the sup norm. Note that X can be isometrically imbedded in $L(X)$.

§4 ABSOLUTE EXTENSORS

For every category where the notion of subobject is defined, one can consider absolute extensors. An object Y in Category \mathcal{C} is an *absolute extensor* in \mathcal{C} , $Y \in AE(\mathcal{C})$, if any other object X and subobject $A \subset X$ and a morphism $\phi : A \rightarrow Y$ there is an extension $\bar{\phi} : X \rightarrow Y$.

A subspace of (X, d) in the category \mathcal{A} is a closed subset with the induced metric $(Z, d|_Z)$, $Z \subset X$. What are $AE(\mathcal{A})$ and $AE(\tilde{\mathcal{A}})$? The one point space is not an absolute extensor because unbounded spaces cannot have proper maps to a point.

Let $\mathbf{R}_+^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n \geq 0\}$ be a half space and denote $\mathbf{R}_+ = \mathbf{R}_+^1$.

Theorem 4.1. $\mathbf{R}_+ \in AE(\mathcal{A})$ and $\mathbf{R}_+ \in AE(\tilde{\mathcal{A}})$.

Proof. First we consider the case $\tilde{\mathcal{A}}$. Let $A \subset X$ be a closed subset and let $\phi : A \rightarrow \mathbf{R}_+$ be a morphism with the coarse Lipschitz constants λ and s . Let $\|\phi(x)\| \geq c\|x\| - b$. We take $c \leq \lambda$. We choose 0 as the base point in \mathbf{R}_+ . Using the idea of [CGM] we define not necessarily continuous extension ϕ' of ϕ by transfinite induction. Enumerate points of $X \setminus A$. Assume that ϕ' is already defined on a set $B_\alpha = A \cup (\cup_{\beta < \alpha} \{x_\beta\})$ with the same constants λ, s, c and b . In order to maintain the asymptotical Lipschitz inequality we should take the point x_α to a point lying in all intervals $I_x = [\phi'(x) - \lambda d(x, x_\alpha) - s, \phi'(x) + \lambda d(x, x_\alpha) + s]$ for $x \in B_\alpha$. Since for $\phi'(x') \leq \phi'(x)$ the inequality $\phi'(x) - \phi'(x') \leq \lambda d(x, x') + s$ and the triangle inequality for x, x' and x_α imply that $\phi'(x) - \lambda d(x, x_\alpha) - s \leq \phi'(x') + \lambda d(x', x_\alpha) + s$, it follows that $I_x \cap I_{x'} \neq \emptyset$. In order to have inequality $\|\phi'(x_\alpha)\| = \phi'(x_\alpha) \geq c\|x_\alpha\| - b$ we should take the point x_α to the interval $J_\alpha = [c\|x_\alpha\| - b, \infty)$. Note that $\phi'(x) + \lambda d(x, x_\alpha) + s \geq c\|x\| - b + \lambda d(x, x_\alpha) + s \geq c(\|x\| + d(x, x_\alpha)) - b + s \geq c\|x_\alpha\| - b + s \geq c\|x_\alpha\| - b$. Hence all intervals $I_x \cap J_\alpha$ are having pairwise nonempty intersections. By Helly Theorem the intersection $I = (\cap I_x) \cap J_\alpha$ is nonempty. Define $\phi'(x_\alpha) \in I$.

By a relative version of Proposition 2.3 there is a continuous extension $\bar{\phi}$ which is in a finite distance with ϕ' .

We cannot apply the above argument in the case of \mathcal{A} , since we do not have a control on properness of extending map. Here we use different argument.

Let $A \subset X$ be a closed subset with the induced metric and let $\phi : A \rightarrow \mathbf{R}_+$ be a proper asymptotically Lipschitz map with constants λ and s . Let R be as in Proposition 2.1 for $\alpha = 2$. We set $m = \lambda R + s$. For all i we define $A_i = \phi^{-1}([0, mi])$ and $B_i = A \setminus A_i$. For any positive function $\xi : E \rightarrow \mathbf{R}_+$, $E \subset X$, we denote by $N_\xi(E) = \cup_{y \in E} B_{\xi(y)}(y)$, the ξ -neighborhood of E . An open ξ -neighborhood of E we denote as $ON_\xi(E)$. If ξ is a constant, say ϵ , then these are ordinary ϵ -neighborhoods. We define a function ξ_i by the formula $\xi_i(y) = \frac{1}{2\lambda}(\phi(y) - mi)$. It is easy to check that $\xi_i > 0$ on B_{i+1} . Denote $O_i = ON_{\xi_i}(B_{i+1})$. By induction we construct a family $\{C_i\}$ of compact subsets of X with the properties:

- (1) $C_i \cap A = A_i$ and $Int(C_i) \cap A = Int_A(A_i)$,
- (2) $N_{\frac{m}{2\lambda}}(C_i) \subset C_{i+1}$,
- (3) $C_i \cap O_i = \emptyset$,
- (4) $N_{\frac{m}{2\lambda}}(C_i) \cap B_{i+1} = \emptyset$

Conditions (3),(4) are technical, they are needed only for a smooth work of induction.

We start with $C_0 = A_0$. Now assume that we have constructed C_0, \dots, C_i satisfying (1)-(4). Since ϕ is a proper map, A_{i+1} is a compact set. There is a compact set C' satisfying (1). Note that $C = C' \cup N_{\frac{m}{2\lambda}}(C_i)$ satisfies (1) as well. Indeed, $C \cap A = (C' \cap A) \cup (N_{\frac{m}{2\lambda}}(C_i) \cap A) = A_{i+1} \cup (N_{\frac{m}{2\lambda}}(C_i) \cap A_{i+1}) \cup (N_{\frac{m}{2\lambda}}(C_i) \cap B_{i+1}) = A_{i+1}$ by virtue of (4). We define $C_{i+1} = C \setminus O_{i+1}$.

To check (1) it suffices to show that $Cl(O_{i+1}) \cap A_{i+1} = \emptyset$. If $x \in Cl(O_{i+1}) \cap A_{i+1}$, then

there is $y \in Cl(B_{i+2})$ with $d(x, y) \leq \xi_{i+1}(y)$. Since $\phi(x) \leq m(i+1)$ and $\phi(y) \geq m(i+2)$, $\phi(y) - \phi(x) \geq m$. Hence $\lambda d(x, y) + s \geq m = \lambda R + s$, i.e. $d(x, y) \geq R$. Therefore, $\phi(y) - \phi(x) < 2\lambda d(x, y)$. On the other hand, $\phi(y) - \phi(x) \geq \phi(y) - m(i+1) = 2\lambda \xi_{i+1}(y) \geq 2\lambda d(x, y)$. The contradiction implies that $Cl(O_{i+1} \cap A_{i+1}) = \emptyset$.

To check (2), it suffices to show that $N_{\frac{m}{2\lambda}}(C_i) \cap O_{i+1} = \emptyset$. Assume the contrary, there is $x \in N_{\frac{m}{2\lambda}}(C_i) \cap O_{i+1}$. Then there are points $z \in C_i$ and $y \in B_{i+2}$ such that $d(x, z) \leq m/2\lambda$ and $d(x, y) < \xi_{i+1}(y)$. Then by the triangle inequality $d(z, y) < \xi_{i+1}(y) + m/2\lambda = \frac{1}{2\lambda}(\phi(y) - mi) = \xi_i(y)$. Hence, $z \in O_i$. Thus, $z \in C_i \cap O_i$ which contradicts with (3).

The condition (3) is satisfied automatically. Check the condition (4). Assume the contrary: there is $x \in N_{\frac{m}{2\lambda}}(C_{i+1}) \cap B_{i+2}$. Then there is $y \in C_{i+1}$ with $d(x, y) \leq m/2\lambda$. On the other hand, by the condition (3) for C_{i+1} we have $d(x, y) \geq \xi_{i+1}(x) = 1/2\lambda(\phi(x) - m(i+1))$. These inequalities imply $m(i+2) \geq \phi(x)$. So, x cannot be in B_{i+2} . Contradiction.

We note that the condition (2) implies that $\cup C_i = X$. We define $\phi(\partial C_i) = mi$. Because of the condition (1) this is a continuous extension of ϕ over $A' = A \cup (\cup_{i=0}^{\infty} \partial C_i)$. Consider the sets $D_i = C_i \setminus Int(C_{i-1})$. Extend the map $\phi|_{D_i \cap A'} : D_i \cap A' \rightarrow [m(i-1), mi]$ to an arbitrary continuous map $\bar{\phi}_i : D_i \rightarrow [m(i-1), mi]$. The union $\bar{\phi} = \cup \bar{\phi}_i$ is a proper continuous map $\bar{\phi} : X \rightarrow \mathbf{R}_+$. Show that $\bar{\phi}$ is asymptotically Lipschitz. Take two points $x, y \in X$. Take the minimal number i such that $x \in C_i$. Similarly let j be the minimal number such that $y \in C_j$. With out loss of generality we may assume that $j = i + k$. Then $|\bar{\phi}(y) - \bar{\phi}(x)| \leq m(k+1)$. By the condition (2) we have $d(x, y) \geq (k-1)m/2\lambda$. Therefore, $k+1 \leq (2\lambda/m)d(x, y) + 2$. Hence, $|\bar{\phi}(y) - \bar{\phi}(x)| \leq 2\lambda d(x, y) + 2m$ for all $x, y \in X$. \square

REMARK 1. In the proof of Theorem 4.1 in the case of \mathcal{A} we switched from a Lipschitz constant λ to 2λ . We note that we can take $\alpha\lambda$ for any $\alpha > 1$.

REMARK 2. The space \mathbf{R}_+ would not be an $AE(\mathcal{A})$ if we take coarse maps in Roe's sense as morphisms in \mathcal{A} . Then the map $\phi : A \rightarrow \mathbf{R}_+$ for $A = \{n^2\} \subset \mathbf{R}$ defined by the formula $\phi(n^2) = 2^n$, being a coarse map, does not have a coarse extension, since no extension of ϕ can be asymptotically Lipschitz.

EXAMPLE 1. \mathbf{R}^n is not an $AE(\mathcal{A})$. It is not an absolute extensor for itself. To see that one can take $X = \mathbf{R}^n$ and $A = \cup S_{2^n}(0)$, the union of spheres of rapidly growing radii, and define a map $f : A \rightarrow \mathbf{R}^n$ as the union of maps $f_n : S_{2^n}(0) \rightarrow S_n(0)$ of degree n . Then f does not have an extension to a proper map.

EXAMPLE 2. A simpler proof that \mathbf{R}^n is not an absolute extensor is the following: there is no proper retraction of \mathbf{R}_+^{n+1} onto \mathbf{R}^n . This proof is good for $\tilde{\mathcal{A}}$ as well.

Lemma 4.2. *The space \mathbf{R}_+^n is isomorphic in $\tilde{\mathcal{A}}$ (and hence in \mathcal{A}) to the n -th power $(\mathbf{R}_+)^n$.*

Proof. There is the natural embedding of $(\mathbf{R}_+)^n$ in \mathbf{R}^n generated by the imbedding $\mathbf{R}_+ \subset \mathbf{R}$. We imbed \mathbf{R}_+^n in \mathbf{R}^n as the halfspace bounded by the hyperplane $A: x_1 + \dots + x_n = 0$ which contains $(\mathbf{R}_+)^n$. We define a morphism $T : (\mathbf{R}_+)^n \rightarrow \mathbf{R}_+^n$ by the following rule. For any line l parallel to the vector $\bar{d} = (1, \dots, 1)$ the restriction T_l of T onto $l \cap (\mathbf{R}_+)^n$

is a translation by vector \bar{w}_l in the direction $-\bar{d}$. The norm $\|\bar{w}_l\|$ we denote by w_l . If $\bar{x} = (x_1, \dots, x_n)$ is a point on l then we define $w_l = \frac{1}{n}(\sum x_i - mn)$ where $m = \min x_i$. We note that the definition of w_l does not depend on the choice of $\bar{x} \in l$. Note that $m = 0$ for a point $\bar{x} \in \partial(\mathbf{R}_+)^n$. Since $\sum_j (x_j - \frac{1}{n}\sum_i x_i) = 0$, the boundary of $(\mathbf{R}_+)^n$ is taken to A by T and hence T is surjective. This implies that T is bijective and T^{-1} is the family of inverse translations $\{-\bar{w}_l\}$. If \bar{x} and \bar{y} are two points lying on lines l and l' , then $\|\bar{w}_l - \bar{w}_{l'}\| = |w_l - w_{l'}| = \frac{1}{n}|\sum x_i - \sum y_i + m'n - mn| \leq \frac{1}{n}\sum |x_i - y_i| + |m' - m|$. Show that $|m' - m| \leq \sum |x_i - y_i|$. Indeed, we may assume that $m < m'$. Let m is achieved on i . Then $|y_i - x_i| = y_i - m \geq m' - m$. Then $\|\bar{w}_l - \bar{w}_{l'}\| \leq \frac{n+1}{n}\sum |x_i - y_i| \leq nd(\bar{x}, \bar{y})$. Therefore $d(T(\bar{x}), T(\bar{y})) \leq (n+1)d(\bar{x}, \bar{y})$. The same Lipschitz constant is good for T^{-1} .

Note that $\|T(x)\| = \|x + \bar{w}_l\| \geq \|x\| - w_l \geq (1 - \frac{1}{\sqrt{2}})\|x\|$. The last inequality is due to the fact that $w_l = \|pr_A x\| \leq \frac{1}{\sqrt{2}}\|x\|$. This implies that T has nonzero norm. By virtue of the inequality $\|T^{-1}(x)\| \geq \|x\|$ the inverse map T^{-1} also has a nonzero norm. \square

Theorem 4.3. $\mathbf{R}_+^n \in AE(\mathcal{A})$ and $\mathbf{R}_+^n \in AE(\tilde{\mathcal{A}})$ for all n .

Proof. According to Lemma 4.2 it suffices to prove that for the n -th power of \mathbf{R}_+ . We fix a basis $\bar{v}_1, \dots, \bar{v}_n$ in \mathbf{R}^n with $\bar{v}_1 = (1, \epsilon, \dots, \epsilon), \dots, \bar{v}_n = (\epsilon, \dots, \epsilon, 1)$ for some small ϵ . We define a projection $p_i : (\mathbf{R}_+)^n \rightarrow \mathbf{R}_+$ onto the i -th factor by the formula $p_i(\bar{z}) = \bar{z} \cdot \bar{v}_i$. Geometrically, p_i is a projection onto x_i -axis, parallel to the plane α_i , where α_i is orthogonal to \bar{v}_i . Note that each p_i is a proper Lipschitz map.

We define a linear isomorphism $p' : \mathbf{R}^n \rightarrow \mathbf{R}^n$ with $p'(\bar{e}_i) = \bar{v}_i$ where $\{\bar{e}_i\}$ is the standard orthonormal basis in \mathbf{R}^n . It is easy to check that $p'(\bar{z}) = \sum p_i(\bar{z})\bar{e}_i$. Denote by $p : (\mathbf{R}_+)^n \rightarrow p'((\mathbf{R}_+)^n) = V$ the restriction of p' onto $(\mathbf{R}_+)^n$. The inverse map $p^{-1} : V \rightarrow (\mathbf{R}_+)^n$ can be extended to a Lipschitz map $q : (\mathbf{R}_+)^n \rightarrow (\mathbf{R}_+)^n$. To demonstrate that first we note that V is the union of rays in \mathbf{R}^n emanated from the origin through a simplex $\sigma \subset \Delta^{n-1}$ lying in the standard simplex $\Delta^{n-1} = \{\bar{x} \mid \sum x_i = 1, x_i \geq 0\}$. Moreover the simplex σ is obtained from Δ^{n-1} by a contraction c with the fixed point in the center of Δ^{n-1} . Let $\gamma = c^{-1} : \sigma \rightarrow \Delta^{n-1}$ be the inverse map. We can extend γ to $\xi : \Delta^{n-1} \rightarrow \Delta^{n-1}$ by taking the radial contraction of the collar $\Delta^{n-1} \setminus \sigma$ to $\partial\Delta^{n-1}$. The map ξ can be linearly extended to a map $\beta : (\mathbf{R}_+)^n \rightarrow (\mathbf{R}_+)^n$. It is clear that β is Lipschitz. The restriction $\beta|_V$ is a linear map which takes \bar{v}_i to $(1 + (n-1)\epsilon)\bar{e}_i$. Hence, $\frac{1}{1+(n-1)\epsilon}\beta|_V = p^{-1}$. Then $q = \frac{1}{1+(n-1)\epsilon}\beta$.

Let $\phi : A \rightarrow \mathbf{R}_+^n$ be a morphism in $\mathcal{A}(\tilde{\mathcal{A}})$, where $A \subset Z$. Then $p_i \circ \phi$ is a morphism in $\mathcal{A}(\tilde{\mathcal{A}})$ for any i . By Theorem 4.1 there are extensions $\bar{\phi}_i : Z \rightarrow \mathbf{R}_+$ with constants λ_i, s_i . Consider the map $\psi = q \circ \bar{\phi} : Z \rightarrow (\mathbf{R}_+)^n$ where $\bar{\phi} = (\bar{\phi}_1, \dots, \bar{\phi}_n)$. For $z \in A$ we have $\psi(z) = q(\bar{\phi}_1(z), \dots, \bar{\phi}_n(z)) = p^{-1}(p_1 \circ \phi(z), \dots, p_n \circ \phi(z)) = p^{-1}p\phi(z) = \phi(z)$. Thus, ψ is an extension of ϕ . The map ψ is proper (with nonzero norm), since all $\bar{\phi}_i$ are of that kind. The map ψ is asymptotically Lipschitz with constants λm and s where $m = \max\{\lambda_i\}$, $s = \max\{s_i\}$ and λ is a Lipschitz constant for q . \square

Lemma 4.4. Let A be a closed subset of a proper metric space (X, d) and let $g : A \rightarrow \mathbf{R}^n$ be an asymptotically Lipschitz map. Then there is a neighborhood $W \supset A$ with

$\|g(a)\| \leq \lambda(d(a, X \setminus W)) + s$ for some numbers λ and s and for all $a \in A$ which admits an asymptotically Lipschitz extension $\bar{g} : W \rightarrow \mathbf{R}^n$ of g .

Proof. Let $\pi : \mathbf{R}_+^{n+1} \rightarrow \mathbf{R}^n$ be the orthogonal projection. By Theorem 4.3 there is an asymptotically Lipschitz extension $g' : X \rightarrow \mathbf{R}_+^{n+1}$ with constants λ' and s' . We define $\lambda = \sqrt{2}\lambda'$, $s = \sqrt{2}s'$ and $W = (g')^{-1}(V)$ where V is the region under the graph of the function $\|x\|$ on $\mathbf{R}^n \subset \mathbf{R}_+^{n+1}$. Then $\bar{g} = \pi \circ g'$. Let $w \in X \setminus W$, then $\|g(a) - g'(w)\| \geq \frac{1}{\sqrt{2}}\|g(a)\|$. Therefore, $\lambda'd(a, w) + s' \geq \frac{1}{\sqrt{2}}\|g(a)\|$. Hence, $\lambda(d(a, w)) + s \geq \|g(a)\|$ for all $w \in X \setminus W$. Thus, $\lambda(d(a, X \setminus W)) + s \geq \|g(a)\|$ for all $a \in A$. \square

REMARK 4. Lemma 4.4 holds true if one replace \mathbf{R}^n by \mathbf{R}_+^n .

We recall that a coarsely proper map f is characterized by the property that a preimage $f^{-1}(C)$ of any bounded set is bounded.

Proposition 4.5. *For any coarsely proper (not necessarily continuous) function $f : X \rightarrow \mathbf{R}_+$ of a proper metric space (X, d) there is a proper asymptotically Lipschitz function $q : X \rightarrow \mathbf{R}_+$ with $q \leq f$.*

Proof. Denote $A_k = f^{-1}([0, k])$. Consider an increasing sequence of concentric balls $B_{m_k}(x_0)$ of integral radii, $x_0 \in X$ and $k \in \mathbb{N}$ such that $A_k \subset B_{m_k}(x_0)$. Define $q(x) = k - 2 + \frac{d(x, x_0) - m_{k-1}}{m_k - m_{k-1}}$ for $x \in B_{m_k}(x_0) \setminus B_{m_{k-1}}(x_0)$ and $k > 1$. We set $q(B_{m_1}(x_0)) = 0$. Clearly q is a proper continuous map. Since $f(B_{m_k}(x_0) \setminus B_{m_{k-1}}(x_0)) \geq k - 1$ and $q(B_{m_k}(x_0) \setminus B_{m_{k-1}}(x_0)) \leq k - 1$, we have $q \leq f$. To check the asymptotical Lipschitz condition we take two points $x \in B_{m_k}(x_0) \setminus B_{m_{k-1}}(x_0)$ and $y \in B_{m_l}(x_0) \setminus B_{m_{l-1}}(x_0)$ with $k \geq l$. Then $|q(x) - q(y)| = |k - l + \frac{d(x, x_0) - m_{k-1}}{m_k - m_{k-1}} - \frac{d(y, x_0) - m_{l-1}}{m_l - m_{l-1}}| \leq |k - l| + 2 \leq d(x, y) + 3$. \square

Lemma 4.6. *Let A be a closed subset of a proper metric space (X, d) , let $W \supset A$ be a closed neighborhood and let $g : W \rightarrow \mathbf{R}_+$ be proper asymptotically Lipschitz with $\lambda(d(a, X \setminus W)) + s \geq g(a)$ for some constant λ and s and all $a \in A$. Let $f : X \rightarrow \mathbf{R}_+$ be a coarsely proper map with $g \leq f$. Then there exists a proper asymptotically Lipschitz map $\bar{g} : X \rightarrow \mathbf{R}_+$ with $\bar{g} \leq f$ and $\bar{g}|_A = g$.*

Proof. Since \mathbf{R}_+ is AR, there is an asymptotically Lipschitz extension $g' : X \rightarrow \mathbf{R}_+$ of g . Apply Proposition 4.5 to a proper map $f' = \min\{f, g'\}$ to obtain an asymptotically Lipschitz map $q : X \rightarrow \mathbf{R}_+$ such that $q \leq f$ on X and $q \leq g$ on W . We define $\phi(x) = \frac{d(x, X \setminus W)}{d(x, A) + d(x, X \setminus W)}$. Since $\phi(X \setminus A) = 0$, the function $\bar{g}(x) = q(x) + \phi(x)(g(x) - q(x))$ is well-defined. Since $q \leq \bar{g}$ and $\phi \geq 0$, we have that $\bar{g}(x)$ tends to infinity when x approaches infinity i.e. when $d(x, x_0) \rightarrow \infty$ for some (all) fixed point $x_0 \in X$. Note that $\bar{g}|_A = g$. To complete the proof we verify that \bar{g} is asymptotically Lipschitz. First, we note that

$$|\phi(x) - \phi(y)| = \frac{1}{d(y, A) + d(y, X \setminus W)} \frac{|d(y, A)d(x, X \setminus W) - d(x, A)d(y, X \setminus W)|}{d(x, A) + d(x, X \setminus W)} \leq$$

$$\frac{1}{d(y, A) + d(y, X \setminus W)} (|d(y, A) - d(x, A)|\phi(x) + |d(x, X \setminus W) - d(y, X \setminus W)|(1 - \phi(x)))$$

$$\leq \frac{1}{d(y,A)+d(y,X \setminus W)}(d(x,y)\phi(x) + d(x,y)(1 - \phi(x))) = \frac{1}{d(y,A)+d(y,X \setminus W)}d(x,y).$$

Then $|\bar{g}(x) - \bar{g}(y)| \leq |q(x) - q(y) + \phi(x)g(x) - \phi(x)q(x) - \phi(y)g(y) + \phi(y)q(y)| \leq |q(x) - q(y)| + \phi(x)|g(x) - g(y)| + g(y)|\phi(x) - \phi(y)| + q(y)|\phi(x) - \phi(y)| + \phi(x)|q(x) - q(y)| \leq 2|q(x) - q(y)| + |g(x) - g(y)| + 2g(y)|\phi(x) - \phi(y)|$. Here we use the facts that $q(y) \leq g(y)$ and $\phi(x) \leq 1$. Let λ_1, s_1 and λ_2, s_2 be constants from the definition of the asymptotical Lipschitz property for g and q . Then we can conclude that

$$|\bar{g}(x) - \bar{g}(y)| \leq 2\lambda_2 d(x,y) + 2s_2 + \lambda_1 d(x,y) + s_1 + 2g(y) \frac{1}{d(y,A)+d(y,X \setminus W)} d(x,y) \leq (2\lambda_2 + \lambda_1 + \frac{\lambda d(y,X \setminus W) + s}{d(y,A)+d(y,X \setminus W)}) d(x,y) + s_1 + 2s_2.$$

Let $m = \min\{d(y,A) + d(y,X \setminus W)\}$. Then $\frac{\lambda d(y,X \setminus W) + s}{d(y,A)+d(y,X \setminus W)} \leq (\lambda + s/m)$.

Therefore $|\bar{g}(x) - \bar{g}(y)| \leq (\lambda_1 + 2\lambda_2 + \lambda + s/m) d(x,y) + s_1 + 2s_2$. \square

§5 ANE, HOMOTOPY, DIMENSION

For any closed subset $A \subset X$ of a proper metric space X we define an *asymptotic neighborhood* W of A in \mathcal{A} as a subset of X containing A with the property $\lim_{R \rightarrow \infty} d(A \setminus B_R(x_0), (X \setminus W) \setminus B_R(x_0)) = \infty$ for some (= any) point $x_0 \in X$.

Similarly, in $\tilde{\mathcal{A}}$, a set W should have the property $d(x, X \setminus W) \geq k\|x\|$ for some $k > 0$ for all $x \in A$.

DEFINITION 1 (conventional). An object $Y \in \mathcal{A}$ (or $\tilde{\mathcal{A}}$) is called an *absolute neighborhood extensor*, $Y \in ANE(\mathcal{A})$, if for any object X and any subobject $A \subset X$, for every morphism $f : A \rightarrow Y$ there is an extension $\tilde{f} : W \rightarrow Y$ of f to a morphism of a closed asymptotic neighborhood.

This definition has a flaw. In the category of topological spaces, metric compacta in particular, there is a fact connecting AE and ANE . Namely, X is ANE if and only if the cone over it CX is AE. In category \mathcal{A} this is not the case. Let X be a parabola, lying in \mathbf{R}_+^2 with the induced metric. Since X is a retract of an asymptotic neighborhood in AE -space \mathbf{R}_+^2 , X is a conventional ANE. It is not difficult to show that the imbedding of X in CX cannot be extended over \mathbf{R}_+^2 . If one prefers to consider geodesic metric spaces, he could take a paraboloid with an inner metric.

Note that the implication $CX \in AE \Rightarrow X \in ANE$ always holds.

DEFINITION 2 (categorical). X is ANE_0 if $X \times \mathbf{R}_+ \in AE$.

Clearly $ANE_0(\mathcal{A}) \subset ANE(\mathcal{A})$.

Proposition 5.1. $ANE_0(\tilde{\mathcal{A}}) \subset ANE(\tilde{\mathcal{A}})$.

Proposition 5.2. $\mathbf{R}^n \in ANE_0$ in both categories \mathcal{A} and $\tilde{\mathcal{A}}$.

Proof. The result follows from Theorem 4.3.

To make the analogy with the topological category more visual we denote the half plane \mathbf{R}_+^2 by \mathbb{I} .

Theorem 5.3 (HET). *Let $Y \in ANE(\tilde{\mathcal{A}})$. Let A be a closed subset of X and let $f : i_-(X) \cup A \tilde{\times} \mathbb{I} \rightarrow Y$ be a morphism. Then there is an extension $\bar{f} : X \tilde{\times} \mathbb{I} \rightarrow Y$ to a morphism.*

Proof. Since $Y \in ANE$, there is an extension $g : W \rightarrow Y$ of f to a neighborhood. It implies that $d(x, (X \tilde{\times} \mathbb{I}) \setminus W) \geq \lambda \|x\|$. Therefore one can construct a Lipschitz function $\phi : X \rightarrow \mathbf{R}_+$ extending a function ψ on A , given by the formula $\psi(x) = d(x, x_0) + 1$, such that the region D under the graph of this function lies in W . Here we consider the realization of $X \tilde{\times} \mathbb{I}$ in the space $X \times \mathbf{R}_+$. Consider the map $h : X \times \mathbb{I} \rightarrow D$, defined by taking each interval $\{x\} \times [0, d(x, x_0) + 1]$ linearly to $\{x\} \times [0, \phi(x)]$. Let λ' be a Lipschitz constant of ϕ . Show that h is Lipschitz. Indeed,

$$\begin{aligned} d_{X \times \mathbf{R}_+}(h(x, t), h(x', t')) &= d_X(x, x') + \left| \frac{t}{\|x\|+1} \phi(x) - \frac{t'}{\|x'\|+1} \phi(x') \right| \leq \\ d_X(x, x') &+ \frac{t}{\|x\|+1} |\phi(x) - \phi(x')| + \phi(x') \left| \frac{t}{\|x\|+1} - \frac{t'}{\|x'\|+1} \right| \leq \\ d_X(x, x') &+ \lambda' d_X(x, x') + \frac{\phi(x')}{\|x'\|+1} (t - t') + \frac{t}{(\|x\|+1)(\|x'\|+1)} \left| \|x'\| - \|x\| \right| \leq \\ (\lambda' + 1) d_X(x, x') &+ |t - t'| + d_X(x, x') \leq (\lambda' + 2) (d_X(x, x') + |t - t'|) = \\ (\lambda' + 2) d_{X \times \mathbf{R}_+}((x, t), (x', t')). \end{aligned}$$

Since $\|x\|+1 \geq t$, we have $\|h(x, t)\| = \|x\| + \frac{t}{\|x\|+1} \phi(x) \geq \frac{1}{2}(\|x\|+t) - \frac{1}{2} = \frac{1}{2}\|(x, t)\| - \frac{1}{2}$. Thus, h has nonzero norm.

We define $\bar{f} = g \circ h$. \square

DEFINITION. A *homotopy* between two morphisms $f, g : X \rightarrow Y$ is a morphism $H : X \tilde{\times} \mathbb{I} \rightarrow Y$ such that $H|_{i_-(X)} = f$ and $H|_{i_+(X)} = g$.

The notion of homotopy leads to the notion of homotopy equivalence.

EXAMPLE. As it was shown in [Roe2] \mathbf{R}^n is homotopy equivalent to the n -dimensional hyperbolic space \mathbf{H}^n in \mathcal{A} . They are not homotopy equivalent in $\tilde{\mathcal{A}}$, since \mathbf{R}^n does not admit a Lipschitz degree one map to \mathbf{H}^n with nonzero norm.

DEFINITION. A metric space X is said to be of *bounded geometry*, $X \in BG$, if for every L there is a uniformly bounded cover \mathcal{U} of X with the Lebesgue number $> L$ and of finite multiplicity.

Let X be a metric space of bounded geometry, denote by $d(L)$ the minimal multiplicity $m(L)$, which can be achieved in the above definition, minus one: $d(L) = m(L) - 1$.

DEFINITION 0 (Gromov. [G1]). The maximum $\max\{d(L) \mid L \in \mathbf{R}_+\}$, if exists, is called the *asymptotic dimension* of X and is denoted $as \dim X$.

This is a coarse analog of the Lebesgue covering dimension. In classical topology there are several different definition of dimension which lead to the same result (for nice

class of spaces, say compact metric). We consider here Ostrand's definition, Alexandroff-Urysohn's, and Alexandroff-Hurewicz' (see §1).

DEFINITION 1 (Gromov [G1]). *as dim* $X \leq n$ if for any $L > 0$ there are $n + 1$ L -disjoint families \mathcal{U}_i of uniformly bounded sets, $i = 0, \dots, n$ such that the union $\bigcup \mathcal{U}_i$ forms a cover of X .

This definition agrees with **DEFINITION 0** [G1].

The *width* of a simplex Δ lying in a Banach space $(V, \|\cdot\|)$ is defined as the minimal distance from the barycenter of Δ to a face $\sigma \subset \Delta$. An *asymptotic polyhedron* P is a locally finite polyhedron supplied with an intrinsic metric whose restriction on each simplex has a metric induced from some Banach space (the same for all simplices) such that widths of simplices tend to infinity as one recede from a base point. The latter means that for any $M > 0$ there is a finite subcomplex $K \subset P$ such that the width of any simplex from $P \setminus K$ is greater than M .

Let $\phi : X \rightarrow Y$ be a morphism and let $X, Y \subset Z$. The Alexandroff norm of ϕ is the displacement function $\|\phi\|_A : X \rightarrow \mathbf{R}_+$ defined by the formula $\|\phi\|_A(x) = d_Z(x, \phi(x))$. Let $f : X \rightarrow \mathbf{R}_+$ be a proper function. We say that *the Alexandroff-Gromov norm* of $\phi : X \rightarrow Y$ does not exceed f , $\|\phi\|_{AG} < f$, if there exists a metric space Z and isometric imbeddings $X, Y \subset Z$ such that $\|\phi\|_A(x) < f(x)$. We also define *the Urysohn norm* of $\phi : X \rightarrow Y$ to be majorated by a proper function $f : X \rightarrow \mathbf{R}_+$, $\|\phi\|_U < f$, if for any R there is a compactum $C \subset X$ such that $\text{diam}(\phi^{-1}(B_R(\phi(x)))) < f(x)$ for all $x \in X \setminus C$. It is easy to see that $(2 + \epsilon)\|\phi\|_{AG} > \|\phi\|_U$ for any $\epsilon > 0$.

DEFINITION 2. A metric space X has an asymptotic dimension $\text{as dim}_* X \leq n$ if for any proper function $f : X \rightarrow \mathbf{R}_+$ there is a short map $\phi : X \rightarrow K$ to an n -dimensional asymptotic polyhedron such that $\|\phi\|_U < f$.

One can use here the Alexandroff-Gromov norm in this definition instead of the Urysohn norm.

Let \mathcal{U} be a cover of a metric space X and let $x \in X$. We denote

$$\begin{aligned} L_{\mathcal{U}}(x) &= \sup\{d(x, X \setminus U) \mid U \in \mathcal{U}\}, \\ \text{mesh}_{\mathcal{U}}(x) &= \sup\{\text{diam}(U) \mid U \in \mathcal{U}, x \in U\}, \\ \text{order}_{\mathcal{U}}(x) &= m_{\mathcal{U}}(x) = \text{card}\{U \in \mathcal{U} \mid x \in U\}. \end{aligned}$$

Proposition 5.3. $\text{as dim}_* X \leq n \Leftrightarrow$ for a given proper map $f : X \rightarrow \mathbf{R}_+$ there is a uniformly bounded cover \mathcal{U} of order $\leq n + 1$ with $\lim_{x \rightarrow \infty} L_{\mathcal{U}}(x) = \infty$ and with $\text{mesh}_{\mathcal{U}}(x) < f(x)$ for all $x \in X \setminus C$ for some compact set C .

Proof. 1) Assume that $\text{as dim}_* X \leq n$ and let $f : X \rightarrow \mathbf{R}_+$ be given. By the definition there is a short map $\phi : X \rightarrow K$ to an asymptotic polyhedron with $\|\phi\|_U < f$. We may assume that ϕ is onto. Let $C_R \subset X$ be a compactum from the definition of the inequality $\|\phi\|_U < f$. Consider a filtration $T_1 \subset T_2 \subset \dots$ of K by subcomplexes such that $\phi(C_i) \subset T_i$. One can define a subdivision K' of K such that $\text{mesh}(x) \leq i/2$ for $x \in T_{i+1}$ and with width tending to infinity. This is possible by taking the standard cubification, regular subdivision of cubes and then barycentric triangulation. We define $\mathcal{U} = \{\phi^{-1}(\text{OSt}(v, K')) \mid v \in (K')^{(0)}\}$ where $\text{OSt}(v, K')$ means the open star of vertex v in K' . Assume that $v \in T_{i+1} \setminus T_i$. Note that $\text{diam}\phi^{-1}(\text{OSt}(v, K')) \leq \text{diam}\phi^{-1}(B_i(y)) <$

$f(x)$ for all $y \in OSt(v, K')$ and $x \in \phi^{-1}(y)$. Thus, $mesh_{\mathcal{U}}(x) < f(x)$. The Lebesgue number $L_{\mathcal{U}}(x)$ can be estimated from below as minimum of width of n -simplices Δ with $\Delta \cap \Delta_x \neq \emptyset$ where Δ_x contains $\phi(x)$. Thus $\lim_{x \rightarrow \infty} L_{\mathcal{U}}(x) = \infty$.

2) Let f be given. We may assume that $f(x) \leq \|x\|$ where $\|x\|$ is the distance to the base point x_0 in X . Let $g(t) = \inf\{f(x) \mid x \in X \setminus B_t(x_0)\}$ and let $\bar{f}(x) = \frac{1}{4}g(\|x\| - f(x))$. Note that \bar{f} tends to infinity as x approaches infinity. Consider a uniformly bounded cover \mathcal{U} of order $\leq n + 1$ with $\lim_{x \rightarrow \infty} L_{\mathcal{U}}(x) = \infty$ and with $mesh_{\mathcal{U}}(x) < \bar{f}(x)$ for $x \in X \setminus C$ for some compact C . Let $\phi : X \rightarrow N(\mathcal{U})$ be projection to the nerve of \mathcal{U} . We can take a piecewise Euclidean metric on $N(\mathcal{U})$ which turns $N(\mathcal{U})$ into an asymptotic polyhedron and which makes the map ϕ short. We have to show that for any R for x close enough to infinity the inequality $diam(\phi^{-1}(B_R(\phi(x)))) < f(x)$ holds. First we note that for x close enough to infinity the ball $B_R(\phi(x))$ lies in the star $St(\{V\}, N(\mathcal{U}))$, where $x \in V \in \mathcal{U}$. Hence $\phi^{-1}(B_R(\phi(x))) \subset \cup_{U \cap V \neq \emptyset, U \in \mathcal{U}} U = V^*$. Let $y \in U$, $U \cap V \neq \emptyset$. Then $d(x, y) \leq mesh_{\mathcal{U}}(x) + mesh_{\mathcal{U}}(y) < \bar{f}(x) + \bar{f}(y) = \frac{1}{4}(g(\|x\| - f(x)) + g(\|y\| - f(y))) \leq \frac{1}{4}(f(x) + \inf\{f(z) \mid z \in X \setminus B_{\|y\| - f(y)}(x_0)\})$. The triangle inequality implies that $\|x\| \geq \|y\| - f(y)$. Hence $x \in B_{\|y\| - f(y)}(x_0)$ and hence, $\inf\{f(z) \mid z \in X \setminus B_{\|y\| - f(y)}(x_0)\} \leq f(x)$. We conclude that $d(x, y) < \frac{1}{2}f(x)$. Therefore $diam(V^*) < f(x)$. \square

The method of proof of Theorem 1.1 of [D-K-U] allows to proof the following.

Proposition 5.4. *as dim $_* X \leq as \dim X$ for all X .*

The following definition of a macroscopic dimension is based on the idea of extending of maps to spheres.

DEFINITION 3. A coarse dimension does not exceed n , $\dim^c X \leq n$, if for any closed subspace $A \subset X$, any morphism $f : A \rightarrow \mathbf{R}^{n+1}$ in \mathcal{A} , there is an extension $\bar{f} : X \rightarrow \mathbf{R}^{n+1}$.

Here \mathbf{R}^{n+1} serves as an analog of n -sphere in \mathcal{A} . There reason to believe that this is a proper analog is based on the following. An analog of a point (minimal AE-space) in \mathcal{A} is a halfline \mathbf{R}_+ . Then an analog of 0-dimensional sphere S^0 might be $\mathbf{R} = \mathbf{R}_+ \cup \mathbf{R}_+$. The suspension in this case is a multiplication by \mathbf{R} .

Latter we will prove the inequality $\dim^c X \leq as \dim_* X$.

DEFINITION. A metric space X has a *slow dimension growth* if $\lim_{L \rightarrow \infty} \frac{d(L)}{L} = 0$.

§6 APPROXIMATION BY POLYHEDRA, COHOMOLOGY AND DIMENSION

A proper metric space X of bounded geometry admits so called *anti-Čech approximation* by polyhedra. This is by the definition a direct sequence $\{K_i, g_{i+1}^i : K_i \rightarrow K_{i+1}\}$ with simplicial projections g_{i+1}^i together with short maps $f_i : X \rightarrow K_i$ such that f_{i+1} is homotopic to $g_{i+1}^i \circ f_i$. All simplices in K_i are isomorphic to the standard simplex of size L_i , the metric on K_i is a geodesic metric, induced by this property, and $\lim L_i = \infty$. Moreover, one may assume that there are short projections $f_{i+1}^i : K_i \rightarrow K_{i+1}$ such that $f_{i+1} = f_{i+1}^i \circ f_i$ and g_{i+1}^i are simplicial approximations of f_{i+1}^i . When X is uniformly contractible, maps f_i admit left proper homotopy inverse $p_i : K_i \rightarrow X$ such that f_{i+1}^i and $f_{i+1} \circ p_i$ are properly homotopic (see [Roe1], [H-R] for more details).

Here we consider simplicial complexes of the kind that participate in the definition of an anti-Čech approximation, i.e. simplicial complexes with a geodesic metric such that all simplices are isomorphic to the standard simplex of size L . We refer to L as the mesh of K , $mesh(K)$ and we call such simplicial complexes as *uniform polyhedra*.

Following Roe [Roe1], [H-R], we define anti-Čech homology (coarse homology) of X as $\hat{H}_*(X) = \lim_{\leftarrow} \{H_*(K_i)\}$. One can take homologies with infinite locally finite chains, as the result he will get Roe's exotic homology $\hat{H}_*^{lf}(X) = HX_*(X)$ as in [H-R]. This definition is dual to that for classical Čech homology and cohomology for compact metric spaces. We recall that in the case of compacta the Čech homology generally does not behave nicely. It is not exact. There is an exact homology theory called the Steenrod homology. If a compactum Y is the limit of an inverse sequence of polyhedra $\{L_i, q_i^{i+1}\}$, the k -th Steenrod homology can be defined by the formula $H_k^s(Y) = H_{k+1}^{lf}(T)$ where T is the telescope generated by the inverse sequence. The Steenrod theory has a flaw on its own: it is not continuous. So both theories are needed. Note when the coefficient group is a field, then these two theories coincide.

In the large scale topology we also form a telescope T out of the direct sequence $\{K_i, g_{i+1}^i\}$ giving an Anti-Čech approximation to a metric space X . Dual to the Steenrod homology we define the Roe cohomology as $H_R^k(X) = H^k(T)$. To define Roe's cohomology with compact support we consider the telescope T^α of one point compactifications αK_i of K_i generated by maps g_{i+1}^i . Then $H_{R,c}^k(X) = H^k(T^\alpha)$. In Roe's notations $H_{R,c}^k(X) = HX^k(X)$. There is a short exact sequence [Roe1]

$$0 \rightarrow \lim_{\leftarrow} H_c^{k-1}(K_i) \rightarrow H_{R,c}^k(X) \rightarrow \hat{H}_c^k(X) \rightarrow 0.$$

A closed subset $A \subset X$ is *closed in category \mathcal{A}* if $X \setminus A$ is an asymptotic neighborhood of some set in X . For every such A the pair (X, A) has an anti-Čech approximation $\{(K_i, L_i), g_{i+1}^i\}$. Similarly one can define homologies and cohomologies of a pair $\hat{H}_i^{lf}(X, A)$, $\hat{H}_c^i(X, A)$ and $\hat{H}_{R,c}^i(X, A)$.

Proposition 6.1. *For any categorically closed subset $A \subset X$ there are exact sequences of a pair (X, A)*

$$1) \dots \rightarrow \hat{H}_i^{lf}(A) \rightarrow \hat{H}_i^{lf}(X) \rightarrow \hat{H}_i^{lf}(X, A) \rightarrow \hat{H}_{i-1}^{lf}(A) \rightarrow \dots,$$

$$2) \dots \leftarrow H_{R,c}^i(A) \leftarrow H_{R,c}^i(X) \leftarrow H_{R,c}^i(X, A) \leftarrow H_{R,c}^{i-1}(A) \leftarrow \dots.$$

Proof. Note that $\hat{H}_k^{lf}(X) = \lim_{\leftarrow} H_k^{lf}(K_i) = \lim_{\leftarrow} H_k^s(\alpha K_i) = \lim_{\leftarrow} H_k^s(T_i^\alpha) = H_k^s(T^\alpha)$, where T_i^α is a finite part of the telescope T^α up to αK_i . Then both exact sequences follow from Steenrod homology and cohomology exact sequences of the pair (T_X^α, T_A^α) .

The inclusion of αX into a telescope αT_X generates homomorphisms $c_* : H_*^{lf}(X) \rightarrow \hat{H}_*^{lf}(X)$ and $c^* : H_{R,c}^*(X) \rightarrow H_c^*(X)$.

Theorem 6.2[H-R],[Roe1]. *If X is uniformly contractible, then c_* and c^* are isomorphisms.*

In the relative case we have the following

Theorem 6.3. *If X is uniformly contractible, then for any categorically closed subset $A \subset X$ there is the equality*

$$1) \hat{H}_*^{lf}(X, A) = \lim_{\rightarrow k} H_*^{lf}(X, N_k(A))$$

and the exact sequence

$$2) 0 \rightarrow \lim_{\leftarrow k} H_c^{*-1}(X, N_k(A)) \rightarrow H_{R,c}^*(X, A) \rightarrow \lim_{\leftarrow k} H_c^*(X, N_k(A)) \rightarrow 0,$$

where $N_k(A)$ is the closed k -neighborhood of A .

Proof. Let $j_{k+1}^k : N_k(A) \rightarrow N_{k+1}(A)$ denote the inclusion. We consider the telescope S formed by these inclusions and let S^α be a corresponding telescope of the one point compactifications $\alpha N_k(A)$. There is a natural inclusion $S^\alpha \subset \alpha X \times \mathbf{R}_+$. We show that the pair $(\alpha X \times \mathbf{R}_+, S^\alpha)$ is proper homotopy equivalent to the pair (T_X^α, T_A^α) . We may assume that an anti-Čech approximation is chosen such that $f_i(N_i(A)) \subset L_i$. We define a map $f : X \times \mathbf{R}_+ \rightarrow T_X$ as the union $\cup f_i$ on the set $X \times \mathbf{N}$ and extend it over the rest of $X \times \mathbf{R}_+$ by means of homotopies between f_{i+1}^i and g_{i+1}^i . For every i there is $k(i)$ such that $p_i(L_i) \subset N_{k(i)}(A)$ and $p_i \circ f_i|_{N_{k(i-1)}(A)}$ is homotopic to the identity $id_{N_{k(i-1)}(A)}$ in $N_{k(i)}(A)$. Then we define $p : T_X \rightarrow X \times \mathbf{R}_+$ as an extension of the union of maps $p_i : K_i \rightarrow X \times k(i)$. Clearly f and p define a proper homotopy equivalence between $X \times \mathbf{R}_+$ and T_X . Easy verification shows that they also define a proper homotopy equivalence of pairs $(X \times \mathbf{R}_+, S)$ and (T_X, T_A) . Then they induce a proper homotopy equivalence of pairs (T_X^α, T_A^α) and $(\alpha X \times \mathbf{R}_+, S^\alpha)$. Then $\hat{H}_i^{lf}(X, A) = H_i^s(T_X^\alpha, T_A^\alpha) = H_i^s(\alpha X \times \mathbf{R}_+, S^\alpha) = \lim_{\rightarrow} H_i^{lf}(X, N_k(A))$ and $H_{R,c}^*(X, A) = H^*(T_X^\alpha, T_A^\alpha) = H^*(\alpha X \times \mathbf{R}_+, S^\alpha)$. Then by Milnor's formula

$$0 \rightarrow \lim_{\leftarrow k} H_c^{*-1}(X, N_k(A)) \rightarrow H_{R,c}^*(X, A) \rightarrow \lim_{\leftarrow k} H_c^*(X, N_k(A)) \rightarrow 0,$$

□

The above definitions of anti-Čech homology and anti-Čech and Roe cohomologies works well for any generalized homology and cohomology theory: $\hat{M}_*^{lf}(X, A) = M_*^s(T_X^\alpha, T_A^\alpha)$, $M_{R,c}^*(X, A) = M^*(T_X^\alpha, T_A^\alpha)$, and $\hat{M}^*(X, A) = \lim_{\leftarrow} M^*(K_i, L_i)$. The statements 6.1-6.3 also hold true for generalized homology (cohomology) case.

DEFINITION. The asymptotical cohomological dimension of a proper metric space $X \in BG$ with respect to coefficient group G is

$$\text{as dim}_G X = \sup\{n \mid H_{R,c}^n(X, A; G) \neq 0, \text{ for some } A\}.$$

Theorem 6.4. *For uniformly contractible space X the inequalities $\text{as dim}_G X \leq \dim_G X \leq \dim X$ hold, where \dim_G is the ordinary cohomological dimension and \dim is the classical covering dimension.*

Proof. Follows from Theorem 6.3.

Gromov's Lemma 7.1 implies the following

Theorem 6.5. *A space X has $\text{asdim } X \leq n$ if and only if X admits an anti-Čech approximation by n -dimensional simplicial complexes.*

§7 HIGSON CORONA

Let $f : X \rightarrow \mathbf{R}$ be a continuous function on a metric space X and let $B_R(x)$ be a ball centered at x of radius R . The R -variation $\text{Var}_R f(x)$ of f is the number $\sup_{y \in B_R(x)} |f(x) - f(y)|$. Let $C_h(X)$ be a family of all bounded continuous functions on X with the property $\lim_{x \rightarrow \infty} \text{Var}_R f(x) = 0$ for every R . The *Higson compactification* of X is the closure of

$X = (\Pi_{f \in C_h(X)})f(X) \subset \mathbf{R}^{C_h(X)}$ in $\mathbf{R}^{C_h(X)}$. The remainder $\nu X = \bar{X} \setminus X$ is called the *Higson corona*. We note that the Higson corona is a covariant functor $\nu : \mathcal{C} \rightarrow \text{Comp}$ from the coarse category to the category of compact Hausdorff spaces [Roe1].

A piecewise Euclidean (PE) complex of *mesh* D is a simplicial complex K whose simplices are isomorphic to the standard simplex of diameter D and the metric on K is the geodesic metric induced by the metrics on simplices. A map $f : X \rightarrow Y$ between metric spaces is called a *short map* if $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$. A map $f : X \rightarrow Y$ is called *uniformly cobounded* if for any R there is a constant C such that the diameter of the preimage $f^{-1}(B_R(y))$ does not exceed C for any point $y \in Y$. We recall that $\text{asdim } X \leq n$ means for arbitrarily large $L > 0$ there is a uniformly bounded open cover of X of multiplicity $\leq n + 1$ with the Lebesgue number $> L$.

Lemma 7.1 [G1]. *For a proper metric space X the following two conditions are equivalent:*

- (1) $\text{asdim } X \leq n$;
- (2) *For any $D > 0$ there is a uniformly cobounded short proper map $f : X \rightarrow K$ to a PE complex of mesh D and dimension n .*

Theorem 7.2. *If $\text{asdim } X < \infty$, then $\text{asdim } X = \dim \nu X$.*

Proof. By [D-K-U], we have $\text{asdim } X \geq \dim \nu X$.

Let $\text{asdim } X = m$, we show that $\dim \nu X \geq m$. By Lemma 7.1 there is a sequence of short maps $\phi_i : X \rightarrow K_i$ with $\dim K_i = m$ and $D_i = \text{mesh}(K_i) \rightarrow \infty$. We define the Lipschitz constant of a family of maps $S = \{f : X \rightarrow Y\}$ as $L(S) = \inf_{f \in S} \{\lambda \mid d_Y(f(x), f(x')) \leq \lambda d_X(x, x')\}$. If this number is not defined we set $L(S) = \infty$. Let $g : X \rightarrow B^n$ be an inessential map, i.e. a map which admits a sweeping $q : X \rightarrow \partial B^n$ with $q|_{g^{-1}(\partial B^n)} = g|_{g^{-1}(\partial B^n)}$. Let $S(g)$ denote the set of all continuous sweepings of g . Fix a base point $x_0 \in X$ and define $\lambda_i^r = \sup\{L(S(\phi_i|_{\phi_i^{-1}(\Delta^m)})) \mid \Delta^m \subset K_i \setminus \text{Int} B_r(\phi_i(x_0))\}$. Let $\lambda_i = \overline{\lim_{r \rightarrow \infty} \lambda_i^r}$.

Show that the sequence $\frac{D_i}{\lambda_i}$ is bounded. Assume not, then there is a subsequence with $\frac{D_{i_k}}{\lambda_{i_k}} > k$. This implies the inequality $\frac{D_{i_k}}{\lambda_{i_k}^{r_k}} > k$ for some r_k . Hence $\lambda_{i_k}^{r_k} < \infty$ and therefore the maps $\phi_{i_k}|_{\phi_{i_k}^{-1}(\Delta^m)} : \phi_{i_k}^{-1}(\Delta^m) \rightarrow \Delta^m$ are inessential for all m -simplices $\Delta^m \subset K_{i_k} \setminus \text{Int} B_{r_k}(\phi_{i_k}(x_0))$. For every such simplex $\Delta^m \subset K_{i_k}$ we consider a sweeping

$\psi_{\Delta^m} : \phi_{i_k}^{-1}(\Delta^m) \rightarrow \partial\Delta^m$ with the Lipschitz constant equal to $L(S(\phi_{i_k} |_{\phi_{i_k}^{-1}(\Delta^m)}))$. The union of such sweepings defines a uniformly cobounded map $\psi_k : X \rightarrow K_{i_k}^{(m-1)} \cup L_k$ where L_k is the star neighborhood of $B_{r_k}(\phi_{i_k}(x_0))$. Let $p : K_{i_k}^{(m-1)} \cup L_k \rightarrow M_k$ be the simplicial map induced by collapsing L_k to a point. Then $\dim M_k = m - 1$. Moreover the map p is short if we consider PE structure on M_k of mesh D_{i_k} . If we rescale the metric on M_k multiplying it by $\frac{1}{\lambda_{i_k}^r}$, then the composition $p \circ \psi_k$ will be a short map as well. Note that $\text{mesh}(M_k) = \frac{D_{i_k}}{\lambda_{i_k}^r} > k$. Lemma 7.1 implies that $\text{asdim} X \leq m - 1$, which contradicts the assumption.

Let $\frac{D_i}{\lambda_i} < b$ for all i . By induction we define a sequence of m -simplices $\Delta_i^m \subset K_i$ such that the sets $A_i = \phi_i^{-1}(\Delta_i^m)$ are disjoint and $L(S(\phi_i |_{\phi_i^{-1}(\Delta_i^m)})) \geq \frac{D_i}{4b}$. If $\Delta_1, \dots, \Delta_i$ are already defined, we take r_0 such that $\bigcup_{l=1}^i A_l \subset \phi_{i+1}^{-1}(B_{r_0}(\phi_{i+1}(x_0)))$. Then there is $r > r_0$ with $\lambda_{i+1}^r > \frac{D_{i+1}}{2b}$. Take $\Delta_{i+1}^m \subset K_{i+1} \setminus \text{Int} B_r(\phi_{i+1}(x_0))$ with $L(S(\phi_{i+1} |_{\phi_{i+1}^{-1}(\Delta_{i+1}^m)})) \geq \frac{D_{i+1}}{4b}$.

Let $p_i : \Delta_i^m \rightarrow \Delta^m$ be a linear map to the standard unit simplex. Thus, p_i is D_i -contraction. Let $A = \bigcup_{i=1}^\infty A_i$. Consider a map $f : A \rightarrow \Delta^m$, defined by the formula $f = \bigcup_{i=1}^\infty p_i \circ \phi_i |_{A_i}$. Since D_i tends to infinity, the variation $\text{Var}_R f(x)$ tends to zero for any given R . Hence there is an extension $\bar{f} : \bar{A} = A \cup \nu A \rightarrow \Delta^m$. We show that the restriction $\bar{f} |_{\nu A}$ is an essential map. Assume that it is inessential, let $f' : \nu A \rightarrow \partial\Delta^m$ be its sweeping. Denote $\partial A_i = \phi_i^{-1}(\partial\Delta^m)$. Consider the map $g = f' \cup f |_{\bigcup_{i=1}^\infty \partial A_i} : \bigcup \partial A_i \cup \nu A \rightarrow \partial\Delta^m$. Since $\bigcup \partial A_i \cup \nu A$ is a closed subset of \bar{A} , there is an extension $\bar{g} : N \rightarrow \partial\Delta^m$ of g over a neighborhood. Then $A_i \subset N$ for large enough i . Let $g_i = \bar{g} |_{A_i}$. Since the map \bar{g} is extendible over νA , we have $\lim_{i \rightarrow \infty} L(\{g_i\}) = 0$. On the other hand, $L(\{g_i\}) \geq L(S(p_i \circ \phi_i |_{\phi_i^{-1}(\Delta_i^m)})) \geq \frac{1}{D_i} \frac{D_i}{4b} = \frac{1}{4b}$. The contradiction completes the proof. \square

Lemma 7.3. *If $\text{as dim}_* X < \infty$, then $\text{as dim}_* X = \dim \nu X$*

Proof. First we give one more reformulation of the definition asdim_* . Namely, $\text{asdim}_* X \leq m \iff$ for any proper function $f : X \rightarrow \mathbf{R}_+$ there is a map $\phi : X \rightarrow K$ to a locally finite piecewise euclidean simplicial complex of mesh one and of dimension $\dim K = m$ such that $\lim_{\Delta} L(\{\phi |_{\phi^{-1}(\Delta)}\}) = 0$ and $\text{diam} \phi^{-1}(\Delta) < \min_{x \in \phi^{-1}(\Delta)} f(x)$ for all simplices $\Delta \subset K \setminus C$ for some compact set C . We use notations taken from the proof of Theorem 7.2, i.e. $L(\{\phi |_{\phi^{-1}(\Delta)}\})$ is the minimal Lipschitz number for the map $\phi : \phi^{-1}(\Delta) \rightarrow \Delta$. Also we will write $\phi \prec f$ for the above condition on fibers of ϕ . It is not difficult to derive this equivalence from Proposition 5.3.

Let $\text{as dim}_* X = m$, show that $\dim \nu X \geq m$.

Claim: There exist a proper map $\phi : X \rightarrow K$ onto PE complex of $\dim K = m$ with mesh one and a sequence of m -simplices Δ_k^m with $L(S\{\phi |_{\phi^{-1}(\Delta_k^m)}\}) > a$ for some number $a > 0$. Indeed, if we assume the contrary then for every proper monotone function $f : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, $f \leq \frac{1}{2}t$, there is a map $\phi : X \rightarrow K$ to a complex K as above with $\phi \prec \frac{1}{2}f(\frac{1}{2}\|\cdot\|)$. Then for every simplex $\Delta^m \subset K$ we take a sweeping ψ_{Δ^m} with the lowest

possible Lipschitz number. Then the union $\psi = \cup_{\Delta \in K} \psi_{\Delta^m} : X \rightarrow K^{m-1}$ will be a map to a PE complex of dimension $m-1$ and of mesh one with $\lim_{\sigma \subset K^{(m-1)}} L(\{\psi|_{\psi^{-1}(\sigma)}\}) = 0$. Note that $\psi^{-1}(\sigma) \subset \cup_{\sigma \subset \Delta} \phi^{-1}(\Delta)$. Then $\text{diam} \psi^{-1}(\sigma) \leq 2 \max_{\sigma \subset \Delta} \text{diam} \phi^{-1}(\Delta) \leq 2 \max_{\sigma \subset \Delta} \min\{\frac{1}{2}f(\frac{1}{2}\|x\|) \mid x \in \phi^{-1}(\Delta)\} \leq \max\{f(\frac{1}{2}\|x\|) \mid x \in \cup_{\sigma \subset \Delta} \Delta\} = M$. Assume that $M = f(\frac{1}{2}\|x_1\|)$ for $x_1 \in \cup_{\sigma \subset \Delta} \Delta$. Let $m = \min_{x \in \psi^{-1}(\sigma)} f(\|x\|) = f(\|x_2\|)$. Since $\text{diam} \psi^{-1}(\sigma) \leq M$, it follows that $\|x - 2\| \geq \|x_1\| - M$. Since $f(t) \leq \frac{1}{2}t$, we have $\|x_2\| \geq \frac{1}{2}\|x_1\|$. Since f is monotone, we obtain $m \geq M$. Thus, $\text{diam} \psi^{-1}(\sigma) < \min_{x \in \phi^{-1}(\sigma)} f(\|x\|)$. Since f is arbitrary enough, we have that $\text{asdim}_* X \leq m-1$. Contradiction.

Assume that $\phi : X \rightarrow K$ and $\{\Delta_k^m\}$ as above. Denote $A_k = \phi^{-1}(\Delta_k^m)$, and $A = \cup A_k$. Let $p_k : \Delta_k^m \rightarrow \Delta^m$ be the identity map onto the standard simplex. The union map $g = \cup_k p_k \circ \phi|_{\text{mid}_{A_k}} : A \rightarrow \Delta^m$ can be extended to a map $\bar{g} : A \cup \nu A \rightarrow \Delta^m$. Show that $\bar{g}|_{\nu A}$ is essential. Assume the contrary: there is sweeping $g' : \nu A \rightarrow \partial \Delta^m$. Then g' is extendible to a sweeping $\tilde{g} : \cup_{k \geq l} A_k \cup \nu A \rightarrow \partial \Delta^m$. In this case $L(S\{\phi|_{A_k}\}) \rightarrow 0$. That contradicts with the inequality $L(S\{\phi|_{A_k}\}) > a > 0$. \square

Lemma 7.3 implies the following

Theorem 7.4. *Either $\text{asdim}_* X = \dim \nu X$ or $\text{asdim}_* X = \text{asdim} X$.*

Proposition 7.5. *Let $f_n : X \rightarrow \mathbf{R}_+$ be a sequence of coarsely proper functions on a proper metric space with $f_n|_W \geq g$ for some function $g : W \rightarrow \mathbf{R}$, given on a closed subset $W \subset X$, and for all n . Then there is a sequence of bounded subsets $A_n \subset X$ and a coarsely proper function $f : X \rightarrow \mathbf{R}_+$ with $f|_W \geq g$ such that $f|_{A_n} \leq n$ and $f|_{X \setminus A_n} \leq f_n$.*

Proof. Let $\bar{g} : X \rightarrow \mathbf{R}_+$ be an extension of g with the property $\bar{g} \leq f_n$ for all n . Define $A_n = \cup_{i=1}^{i=n+1} f_i^{-1}([0, n])$ and $f(x) = \min\{\max\{n-1, \bar{g}(x)\}, f_n(x)\}$ for $x \in A_n \setminus A_{n-1}$. Note that $f_n(A_n \setminus A_{n-1}) > n-1$ by virtue of the definition of A_{n-1} . Hence $f(A_n \setminus A_{n-1}) \geq n-1$. This implies that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. Hence f is coarsely proper. If $x \in A_n$, then $x \in A_k \setminus A_{k-1}$ for some $k \leq n$. Then $f(x) = \min\{\max\{k-1, \bar{g}(x)\}, f_k(x)\} \leq \max\{k-1, \bar{g}(x)\}$. Since $x \in A_k$, there is $i \leq k+1$ such that $x \in f_i^{-1}([0, k])$. Hence $\bar{g}(x) \leq f_i(x) \leq k$. Thus, $f(x) \leq k \leq n$. If $x \in X \setminus A_n$, then $x \in A_m \setminus A_{m-1}$ for some $m > n$. Since x does not belong to A_{m-1} , we have $f_n(x) > m-1$. Since $f_n(x) \geq \bar{g}(x)$, it follows that $f_n(x) \geq \max\{m-1, \bar{g}(x)\}$. Therefore $f_n(x) \geq f(x) = \min\{\max\{m-1, \bar{g}(x)\}, f_m(x)\}$. Clearly, $f(x) \geq g(x)$. \square

Theorem 7.6. *Let X be a proper metric space. Then the following two conditions are equivalent*

- 1) $\dim^c X \leq n$;
- 2) $\dim \nu X \leq n$.

Proof. Assume that $\dim^c X \leq n$. Let $\phi : C \rightarrow S^n$ be a map of a closed subset of νX to the unit n -sphere. There is an extension $\phi' : V \rightarrow S^n$ of ϕ over a closed neighborhood in $\bar{X} = X \cup \nu X$. Then $\text{Var}_R \phi'(x) \rightarrow 0$ as $x \rightarrow \infty$ for any fixed R . We define proper functions $f_n(x) = \frac{1}{\text{Var}_n \phi'(x)}$, $n \in \mathbf{N}$. Apply Proposition 7.5 to the sequence f_n with g

equal the constant zero function to obtain a coarsely proper function $f : X \rightarrow \mathbf{R}_+$ and a filtration $X = \cup A_n$ such that $f(A_n) \leq n$ and $f|_{X \setminus A_n} \leq f_n$.

By Proposition 4.5 there is an asymptotically Lipschitz function $q : X \rightarrow \mathbf{R}_+$ with $q \leq f$ with Lipschitz constants λ and s . We define a map $g : X \cap V \rightarrow \mathbf{R}^{n+1}$ by the formula $g(x) = (q(x), \phi'(x))$ in the polar coordinates. We check that g is asymptotically Lipschitz. We denote by $\alpha(z_1, z_2)$ the angle between z_1 and z_2 . By the Law of Cosines we have

$$\|g(x) - g(y)\|^2 = q(x)^2 + q(y)^2 - 2q(x)q(y)\cos(\alpha(\phi'(x), \phi'(y))) \leq (q(x) - q(y))^2 + q(x)q(y)(\alpha(\phi'(x), \phi'(y)))^2.$$

Let $n - 1 \leq d(x, y) \leq n$. Since $q(x) \leq f_n(x)$ and $q(y) \leq f_n(y)$ on $X \setminus A_n$, we have

$$q(x)q(y)(\alpha(\phi'(x), \phi'(y)))^2 \leq f_n(x)f_n(y)(\alpha(\phi'(x), \phi'(y)))^2 = \frac{\alpha(\phi'(x), \phi'(y))}{\text{Var}_n \phi'(x)} \frac{\alpha(\phi'(x), \phi'(y))}{\text{Var}_n \phi'(y)} \leq 1.$$

For $x \in A_n$ we have $n \geq f(x) \geq q(x)$. Hence,

$$q(x)q(y)(\alpha(\phi'(x), \phi'(y)))^2 \leq n(n + \lambda d(x, y) + s)\pi^2.$$

$$\text{Therefore } \|g(x) - g(y)\|^2 \leq (\lambda d(x, y) + s)^2 + \pi^2((\lambda + 1)d(x, y) + s + 1)^2 \leq$$

$$(\lambda d(x, y) + s + \pi((\lambda + 1)d(x, y) + s + 1))^2.$$

$$\text{Hence } \|g(x) - g(y)\| \leq (\lambda + \pi(\lambda + 1))d(x, y) + (s + \pi(s + 1)).$$

$$\text{Similarly if } y \in A_n, \text{ then } \|g(x) - g(y)\| \leq (\lambda + \pi(\lambda + 1))d(x, y) + (s + \pi(s + 1)).$$

By the assumption there is an asymptotically Lipschitz extension $\bar{g} : X \rightarrow \mathbf{R}^{n+1}$ of g . Hence \bar{g} can be (uniquely) extended over Higson compactifications to $\tilde{g} : \bar{X} \rightarrow \overline{\mathbf{R}^{n+1}}$. Consider the restriction $g' = \tilde{g}|_{\bar{X} \setminus \tilde{g}^{-1}(0)} : \bar{X} \setminus \tilde{g}^{-1}(0) \rightarrow \overline{\mathbf{R}^{n+1}} \setminus \{0\}$. Let $\eta : \mathbf{R}^{n+1} \setminus \{0\} \rightarrow S^n$ be the radial projection and let $\bar{\eta}$ denote the extension to the Higson corona $\bar{\eta} : \overline{\mathbf{R}^{n+1}} \setminus \{0\} \rightarrow S^n$. Note that $\eta \circ g'$ restricted to $V \cap X$ coincides with ϕ' . Hence $\bar{\eta} \circ g'|_C = \phi$.

Now let $\dim \nu X \leq n$. Let $q' : A \rightarrow \mathbf{R}^{n+1}$ be a proper asymptotically Lipschitz map of a closed subset A of X . By Lemma 4.4 there is an asymptotically Lipschitz extension $q : W \rightarrow \mathbf{R}^{n+1}$ over a closed neighborhood W of A with the property $\|q(a)\| \leq \lambda d(a, X \setminus W) + s$ for some numbers λ, s . Since q is asymptotically Lipschitz there is an extension to the Higson compactifications $\bar{q} : \bar{W} \rightarrow \overline{\mathbf{R}^{n+1}}$. Let $\xi : \overline{\mathbf{R}^{n+1}} \rightarrow \mathbf{R}^{n+1} \cup S^n = B^{n+1}$ be the extension over the Higson compactification of a radial homeomorphism $h : \mathbf{R}^{n+1} \rightarrow \text{Int}(B^{n+1})$ given by the formula: $h(t, \theta) = (\frac{t}{t+1}, \theta)$. We note that $\xi(\nu \mathbf{R}^{n+1}) \subset \partial B^{n+1}$. Since $\dim \nu X \leq n$, there is an extension $\psi : \nu X \rightarrow S^n$ of the map $\xi \circ \bar{q}|_{\nu W} : \nu W \rightarrow S^n$.

Let $\bar{\psi} : X \rightarrow B^{n+1}$ be an extension of $\psi \cup h \circ q$. Consider $g : W \rightarrow \mathbf{R}_+$, defined as $g(x) = 1 + \|q(x)\|$. Note that g is asymptotically Lipschitz. Denote the constants by $\bar{\lambda}$ and \bar{s} .

Let $c_n = n\bar{\lambda} + \bar{s}$. Consider $f_n(x) = \frac{c_n}{\text{Var}_n(\bar{\psi}|_W)(x)} + c_n + 1$ for $x \in X \setminus W$ and $f_n(x) = \frac{c_n}{\text{Var}_n(\bar{\psi}|_W)(x)} + c_n + 1$ for $x \in W$.

Show that $g \leq f_n$ on W . Let

$$\text{Var}_n(\bar{\psi}|_W)(x) = \|h(q(x)) - h(q(y))\| = \left\| \frac{\|q(x)\|}{\|q(x)\|+1} \frac{q(x)}{\|q(x)\|} - \frac{\|q(y)\|}{\|q(y)\|+1} \frac{q(y)}{\|q(y)\|} \right\|.$$

Denote $a = \|q(x)\|$. Since $d(x, y) \leq n$, we have $a - c_n \leq \|q(y)\| \leq a + c_n$. Then

$$\begin{aligned} \text{Var}_n(\bar{\psi}|_W)(x) &\leq \left\| \left(\frac{\|q(x)\|}{\|q(x)\|+1} - \frac{\|q(y)\|}{\|q(y)\|+1} \right) \frac{q(x)}{\|q(x)\|} \right\| + \left\| \frac{\|q(y)\|}{\|q(y)\|+1} \left(\frac{q(x)}{\|q(x)\|} - \frac{q(y)}{\|q(y)\|} \right) \right\| \\ &\leq \frac{c_n}{(a+1)(a-c_n+1)} + \left(1 - \frac{1}{a-c_n+1}\right) \frac{c_n}{a-c_n} = \frac{c_n}{a-c_n} - \frac{c_n(c_n+1)}{(a+1)(a-c_n+1)(a-c_n)} \leq \frac{c_n}{a-c_n} \end{aligned}$$

provided $a \geq c_n$. Therefore $\frac{c_n}{\text{Var}_n(\bar{\psi}|_W)(x)} \geq a - c_n$ if $a - c_n \geq 0$. Hence the inequality $\frac{c_n}{\text{Var}_n(\bar{\psi}|_W)(x)} \geq a - c_n$ always holds. Hence, $f_n(x) = \frac{c_n}{\text{Var}_n(\bar{\psi}|_W)(x)} + c_n + 1 \geq a + 1 = g(x)$. Let f be a coarsely proper function defined by Proposition 7.5 for $\{f_n\}$ and g .

Apply Lemma 4.6 to obtain an asymptotically Lipschitz function $\bar{g} : X \rightarrow \mathbf{R}_+$ with $\bar{g} \leq f$ and with $\bar{g}|_A = g|_A$. Let $\tilde{\lambda}$ and \tilde{s} be its Lipschitz constants. We define a map $\tilde{q} : X \rightarrow \mathbf{R}^{n+1}$ as $\tilde{q}(x) = \bar{\psi}(x)\bar{g}(x)$. Note that if $x \in A$, then $\tilde{q}(x) = \frac{\|q'(x)\|}{\|q'(x)\|+1} \frac{q'(x)}{\|q'(x)\|} (\|q'(x)\|+1) = q'(x)$. So, \tilde{q} is an extension of q' . Show that \tilde{q} is asymptotically Lipschitz. Let $x, y \in X$ be given points. Let $n-1 \leq d(x, y) \leq n$.

Then

$$\begin{aligned} \|\tilde{q}(y) - \tilde{q}(x)\| &\leq \bar{g}(y) \|\bar{\psi}(y) - \bar{\psi}(x) + \bar{\psi}(x) - \frac{\bar{g}(x)}{\bar{g}(y)} \bar{\psi}(y)\| \leq \bar{g}(y) \|\bar{\psi}(y) - \bar{\psi}(x)\| + \frac{|\bar{g}(y) - \bar{g}(x)|}{\bar{g}(y)} \|\bar{\psi}(y)\| \\ &\leq \bar{g}(y) \|\bar{\psi}(y) - \bar{\psi}(x)\| + |\bar{g}(y) - \bar{g}(x)|, \text{ since } 1 \leq \bar{g} \leq f \text{ and } \|\bar{\psi}(y)\| \leq 1. \end{aligned}$$

If $y \in A_n$, then $\bar{g}(y) \leq f(y) \leq n \leq d(x, y) + 1$ and we obtain

$$\|\tilde{q}(y) - \tilde{q}(x)\| \leq 2d(x, y) + 2 + \tilde{\lambda}d(x, y) + \tilde{s}.$$

If $y \in X \setminus A_n$, then $\bar{g}(y) \leq f_n(y) = \frac{n\bar{\lambda} + \bar{s}}{\text{Var}_n(\bar{\psi}|_W)(y)} + n\bar{\lambda} + \bar{s}$ for $y \notin W$. Then

$$\|\tilde{q}(y) - \tilde{q}(x)\| \leq \frac{\|\bar{\psi}(y) - \bar{\psi}(x)\|}{\text{Var}_n(\bar{\psi}|_W)(y)} (n\bar{\lambda} + \bar{s}) + 2(n\bar{\lambda} + \bar{s}) + \tilde{\lambda}d(x, z) + \tilde{s} \leq$$

since $\frac{\|\bar{\psi}(y) - \bar{\psi}(x)\|}{\text{Var}_n(\bar{\psi}|_W)(y)} \leq 1$, we can continue

$$\leq 3(n\bar{\lambda} + \bar{s}) + \tilde{\lambda}d(x, y) + \tilde{s} \leq 3((d(x, y) + 1)\bar{\lambda} + \bar{s}) + \tilde{\lambda}d(x, y) + \tilde{s} = (3\bar{\lambda} + \tilde{\lambda})d(x, y) + (3\bar{\lambda} + 3\bar{s} + \tilde{s}).$$

If $y \in W$ we may assume that $x \in W$ as well. Otherwise we apply the above argument to x instead of y . Then $\frac{\|\bar{\psi}(y) - \bar{\psi}(x)\|}{\text{Var}_n(\psi|_W)(y)} \leq 1$ and the above inequality holds.

□

§8 ON COARSE NOVIKOV CONJECTURES

The following coarse statements imply one or another version of the Novikov conjecture.

1. Gromov's conjecture. *A uniformly contractible manifold of bounded geometry (UC & BG) is (rationally) hypereuclidean (hyperspherical).*

2. Weinberger's conjecture. *The homomorphism $\delta : H^i(\nu X; \mathbf{Q}) \rightarrow H_c^{i+1}(X; \mathbf{Q})$ form the exact sequence of the pair $(\bar{X}, \nu X)$ is an epimorphism.*

3. Coarse Baum-Connes conjecture. *Let X be a proper metric space with $X \in UC \& BG$. Then the Roe index map $\mu : K_*^{lf}(X) \rightarrow K_*(C(X))$ is an isomorphism (rational isomorphism or rational monomorphism).*

These conjectures are closely related. The connection between 1 and 2 is discussed in [D-F] and the connection between 2 and 3 in [Ro1],[Ro2].

An open n -manifold X is called (rationally) *hypereuclidean* if there is a morphism $f : X \rightarrow \mathbf{R}^n$ of degree one (nonzero). It is called (rationally) *hyperspherical* if for every $R > 0$ there is a short 'proper map $f : X \rightarrow S^n(R)$ onto the standard sphere of radius R with the degree one (nonzero) [G2]. Here 'proper means that the complement to some compact set in X goes to a single point.

The most general result supporting the above conjectures is the following

Yu's Theorem [Y2]. *If $X \in UC \& BG$ and X admits a coarsely uniform embedding in the Hilbert space, then the Coarse Baum-Connes conjecture holds for X .*

We note that a coarsely uniform embedding is an embedding in the asymptotic category with Roe's morphisms. Thus, $f : X \rightarrow Y$ is a coarsely uniform embedding if there are two functions $\rho_1, \rho_2 : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ tending to infinity such that $\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y))$. For a geodesic metric space X the image $f(X)$ is not necessarily a totally geodesic subspace, then it means that the inverse map is not necessarily coarsely Lipschitz.

This theorem in particular implies the previous Yu's result saying that X with $as \dim X < \infty$ and $X \in UC \& BG$ satisfies the coarse Baum-Connes conjecture. Here we present some results in that direction.

A sequence of points $\{x_n\}$ with a metric d such that $\lim_{n \rightarrow \infty} d(x_n, \{x_1, \dots, x_{n-1}\}) = \infty$ we call a *0-dimensional asymptotic polyhedron*. We note that such space is 0-dimensional in any (asymptotic) sense. A 0-dimensional skeleton of an asymptotic simplex is a 0-dimensional asymptotic polyhedron. The following lemma first was proved by J. Roe (unpublished).

Lemma 8.1. *The coarse Baum-Connes conjecture holds for 0-dimensional asymptotic polyhedra.*

Lemma 8.2. *The coarse Baum-Connes conjecture holds for all finite dimensional asymptotic polyhedra.*

Proof. By induction on dimension of asymptotic polyhedra. The induction starts by virtue of Lemma 8.1. Let K be an n -dimensional asymptotic polyhedron. For every simplex Δ we denote by $\frac{1}{2}\Delta \subset \Delta$ an $\frac{1}{2}$ -homothetic image of Δ with the common center. Let B be the union of $\frac{1}{2}\Delta$ for all n -simplices in K . Let $A = \overline{K \setminus B}$. Note that A is homotopy equivalent in a coarse sense to the $n - 1$ -dimensional skeleton K^{n-1} . The space B is homotopy equivalent to an asymptotic 0-dimensional polyhedron. Note that $C = A \cap B$ is an $n - 1$ -dimensional asymptotic polyhedron. Since the validity of the coarse Baum-Connes conjecture is a coarse homotopy invariant [Ro2], by the induction assumption for spaces A , B and C the coarse Baum-Connes conjecture holds. Then by the Five Lemma and the Mayer-Vietoris sequence it holds for $K = A \cup B$. \square

Theorem 8.3. *Let X be $ANE(\mathcal{A})$ with $\dim_* X < \infty$. Then the coarse Baum-Connes conjecture holds for X .*

The proof of this theorem follows from Lemma 8.2 and next two lemmas.

Lemma 8.4. *Let X be $ANE(\mathcal{A})$ with $\dim_* X < \infty$. Then X is homotopy dominated in \mathcal{A} by an asymptotic polyhedron of finite dimension.*

Proof. Since X is ANE there is a proper map $\alpha : X \rightarrow \mathbf{R}_+$ such that every two α -close morphisms to X are homotopic. Let W be a neighborhood of X in $P(X)$ that admits a retraction $r : W \rightarrow X$ with a Lipschitz constant λ . Let $d(x) = \frac{1}{\lambda} d_{P(X)}(x, W \setminus r^{-1}(B_{\alpha(x)}(x)))$. We take an approximation $\phi : X \rightarrow K_d$ of X by an asymptotic polyhedron $K_d \subset W$ of a finite dimension such that $\|\phi\|_A(x) < d(x)$. Then $r \circ \phi$ is α -close to 1_X . \square

Lemma 8.5. *Let Y homotopy dominate X in \mathcal{A} and assume that the coarse Baum-Connes holds for Y . Then it holds for X .*

Proof. Consider the diagram:

$$\begin{array}{ccc}
 KX_*(X) & \xrightarrow{A_X} & K_*(C^*(X)) \\
 i_* \downarrow & & i'_* \downarrow \\
 KX_*(Y) & \xrightarrow{A_Y} & K_*(C^*(Y)) \\
 r_* \downarrow & & r'_* \downarrow \\
 KX_*(X) & \xrightarrow{A_X} & K_*(C^*(X))
 \end{array}$$

Here homomorphisms i_* , i'_* , r_* and r'_* are induced by morphisms $i : X \rightarrow Y$ and $r : Y \rightarrow X$ such that $r \circ i$ is homotopic to the identity map 1_X . Then i_* is a monomorphism and r'_* is an epimorphism. Since i_* and A_Y are monomorphisms, A_X is a monomorphism. Since r'_* and A_Y are epimorphic, A_X is an epimorphism. \square

Let $q : X \rightarrow \mathbf{R}_+$ be a function, we say that two maps $\psi_0, \psi_1 : Z \rightarrow X$ are q -close if $d(\psi_0(z), \psi_1(z)) < q(\psi_1(z))$. We say that ψ_0, ψ_1 are q -homotopic if there is a homotopy $H : Z \times I \rightarrow X$ joining them such that $\text{diam}(H(\{z\} \times I)) < q(\psi_0(z))$.

Proposition 8.6. *Let $q : X \rightarrow \mathbf{R}_+$ be a function such that $q(x) \leq \frac{1}{2}\|x\|$. Then any two q -close proper maps are proper homotopic.*

Proof. Let $H : Z \times I \rightarrow X$ be a q -homotopy. Let $B_\rho(x_0)$ be the ball of radius ρ . Show that $H^{-1}(B_\rho(x_0))$ is compact. Since $\psi_0 = H|_{Z \times \{0\}}$ is proper, $\psi_0^{-1}(B_{2\rho}(x_0))$ is compact. We show that $H^{-1}(B_\rho(x_0)) \subset \psi_0^{-1}(B_{2\rho}(x_0)) \times I$. For any $z \in Z \setminus \psi_0^{-1}(B_{2\rho}(x_0))$ we have $\text{diam}(H(\{z\} \times I)) < q(\psi_0(z)) < \frac{1}{2}\|\psi_0(z)\| = \frac{1}{2}d(\psi_0(z), x_0)$. Hence $d(x_0, H(\{z\} \times I)) \geq d(x_0, \psi_0(z)) - \text{diam}(H(\{z\} \times I)) \geq d(x_0, \psi_0(z)) - \frac{1}{2}d(\psi_0(z), x_0) = \frac{1}{2}d(\psi_0(z), x_0) > \rho$. \square

Theorem 8.7. *Let X be uniformly contractible proper metric space of bounded geometry with $\dim X < \infty$ and $\text{as dim}_* X < \infty$. Then the monomorphism version of the coarse Baum-Connes conjecture holds true.*

Proof. We construct a morphism $\phi : X \rightarrow K$ to a finite dimensional asymptotic polyhedron K and a map $r : K \rightarrow X$ such that the composition $r \circ \phi$ is proper homotopic to the identity 1_X . Then $r_* \circ \phi_* = \text{id}$. Since for UC and BG spaces $KX_* = K_*^{lf}$, the diagram of Lemma 8.5 turns into

$$\begin{array}{ccc} K_*(X) & \xrightarrow{A_X} & K_*(C^*(X)) \\ \phi_* \downarrow & & \phi'_* \downarrow \\ K_*(Y) & \xrightarrow{A_Y} & K_*(C^*(Y)) \\ r_* \downarrow & & \\ K_*(X) & & \end{array}$$

Then Lemma 8.2 and the argument of Lemma 8.5 would imply that A_X is a monomorphism.

Let $\dim X = m$. Since $X \in UC$ there is a proper monotone function $g_m : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that for any function $q : X \rightarrow \mathbf{R}_+$ any two q -close maps of m -dimensional space to X are $g_m \circ q$ -homotopic.

Let $\text{as dim}_* X = n$. Let $S : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a contractibility function of X . We define $T(t) = \frac{1}{2}S^{-1}(\frac{t}{2})$, $\bar{f} = T^n = T \circ \dots \circ T$ and define $f : X \rightarrow \mathbf{R}_+$ as $f(x) = \bar{f}(\frac{1}{2}g_m^{-1}(\frac{1}{4}\|x\|))$. Let $\phi : X \rightarrow K_f$ be a short map to an asymptotic polyhedron with $\|\phi\|_U < \frac{1}{4}f$. Then we can subdivide K_f to obtain an asymptotic polyhedron with the property that $\text{diam}(\phi^{-1}(\Delta)) < f(x)$ for all $x \in \phi^{-1}(\Delta)$. We can achieve this for Δ lying outside of some compact set. We may (and we will) ignore this restriction in the further argument.

By induction we construct a map $r : K_f^{(k)} \rightarrow X$ of k -dimensional skeleton. For any $v \in K_f^{(0)}$ we define $r(v) \in \phi^{-1}(v)$. Since for every 1-simplex $[u, v]$ with vertices u, v , $\text{diam}\phi^{-1}([u, v]) < f(r(u)) \leq \bar{f}(\frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|)) = T^n(\frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|)) =$

$\frac{1}{2}S^{-1}(\frac{1}{2}T^{n-1}(\frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|)))$, there is an extension of r over $[u, v]$ with $\text{diam}(r([u, v])) < \frac{1}{2}T^{n-1}(\frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|))$. We extend r over $K_f^{(1)}$ in this manner. Note that for any 2-simplex σ spanned by u, v, w , $\text{diam}(r(\sigma^{(1)})) < T^{n-1}(\frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|))$. Therefore there is an extension of r over σ with $\text{diam}(r(\sigma)) < \frac{1}{2}T^{n-2}(\frac{1}{4}\|r(u)\|)$. Finally we will construct a map $r : K_f^{(n)} = K \rightarrow X$ with $\text{diam}(r(\Delta)) < \frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|)$ for any n -simplex Δ where u is a vertex in Δ . Then for any point $x \in X$ the distance $d(x, r\phi(x))$ can be estimated from above as $d(x, r(u)) + d(r(u), r\phi(x)) < f(x) + \frac{1}{2}g_m^{-1}(\frac{1}{4}\|r(u)\|) < f(x) + \frac{1}{2}g_m^{-1}(\frac{1}{4}(\|x\| + f(x))) \leq \frac{1}{2}g_m^{-1}(\frac{1}{4}\|x\|) + \frac{1}{2}g_m^{-1}(\frac{1}{4}\|x\| + \frac{1}{8}\|x\|) \leq g_m^{-1}(\frac{1}{2}\|x\|)$. Here we used the fact that $T(t) < t$ and $g_m^{-1}(t) \leq t$. Hence 1_X and $r \circ \phi$ are $\frac{1}{2}\|x\|$ -homotopic. Proposition 8.6 implies that 1_X and $r \circ \phi$ are proper homotopic. \square

Theorem 8.8. *Let X be a uniformly contractible n -manifold with asymptotic dimension $\text{as dim } X = n$. Then X is hyperspherical.*

Proof. Let Δ be the standard simplex of dimension n . Let c_n be the Lipschitz constant of a map ν taking Δ to a unit hemisphere S_+ homeomorphically except the complement in $\partial\Delta$ to one $n-1$ -dimensional face goes to a point. Let ϵ be given. Consider a short map $\phi : X \rightarrow K$ to n -dimensional polyhedron with mesh $m > \frac{c_n}{\epsilon}$. We consider a sphere S of sufficiently large radius R centered at $s_0 = \phi(x_0)$. Clearly, S separates s_0 and ∞ . Take a smallest subcomplex $N \subset K$ that contains S . For big enough R the complex N separates s_0 and ∞ . Since $\dim K = n$, $n-1$ -dimensional skeleton $N^{(n-1)}$ separates s_0 and ∞ . Let $N' \subset N^{(n-1)}$ be the boundary of the component containing s_0 . Then $M = \phi^{-1}(N')$ separates x_0 and ∞ . We may assume that ϕ is a light simplicial map with respect to some triangulation of X and a small subdivision of K . Then M is a polyhedron. The boundary of the component containing x_0 defines a cycle c in M that generates the $n-1$ -dimensional homology group of $X \setminus \{x_0\}$. Let $C = \text{supp}(c)$. Since $X \in UC$, there is an 'approximate' lift $\alpha : K \rightarrow X$ of ϕ . If R is big enough then $\alpha \circ \phi|_C$ is homotopic to 1_C in $X \setminus \{x_0\}$. Then $(\phi|_C)_*(c) \neq 0$. Hence there is an $(n-1)$ -simplex σ in N' such that $\phi|_{\phi^{-1}(\sigma)}$ is essential. An easy diagram chasing shows that the degree of the homomorphism $\mathbb{Z} = H_{n-1}(C) \rightarrow H_{n-1}(\phi^{-1}(N')) \rightarrow H_{n-1}(\phi^{-1}(N'), \phi^{-1}(N' \setminus \text{Int}(\sigma))) \rightarrow H_{n-1}(N', N' \setminus \text{Int}(\sigma)) = H_{n-1}(\sigma, \partial\sigma) = \mathbb{Z}$ is one. Let W be the star of σ in K , i.e. W is the union of all simplices containing σ . We define a $\frac{c_n}{m}$ -Lipschitz map $\psi : W \rightarrow S^n$ to the unit sphere using the map ν in such a way that all n -simplices $\Delta \supset \sigma$ lying in the component of s_0 go to the lower hemisphere and all other simplices $\Delta \supset \sigma$ are mapped to upper hemisphere. Then the map $\bar{\psi} : X \rightarrow S^n$ defined as $\psi \circ \phi$ on $\phi^{-1}(W)$ and as a constant map on the rest of X is ϵ -Lipschitz of degree one.

Theorem 8.9. *Let X be a metric space with slow dimension growth, then the Coarse Baum-Connes conjecture holds for X .*

The proof is based on Yu's theorem and the following Lemma which is due to Higson-Roe and Yu [H-R2],[Y2]. A probability measure μ on a discrete space Z can be treated as a function $\mu : Z \rightarrow \mathbf{R}_+$ with $\sum_{z \in Z} \mu(z) = 1$, the support $\text{supp}(\mu)$ consists of those $z \in Z$ where μ is nonzero. By $\|\cdot\|_1$ we denote the l_1 -norm on a space of functions.

Lemma 8.10 [H-R2],[Y2]. *Let Z be a discrete space of bounded geometry and assume that there is a sequence of maps $a^n : Z \rightarrow P(Z)$ to probability measures such that*

- (1) *For any n there is R such that $\text{supp}(a^n(z)) \subset \{z' \in Z \mid d(z, z') < R\}$.*
- (2) *For any $K > 0$, $\lim_{n \rightarrow \infty} \sup_{d(z, w) < K} \|a^n(z) - a^n(w)\|_1 = 0$.*

Then Z admits a coarsely uniform embedding into a Hilbert space.

Proof of Theorem 8.9. It suffices to embed an ϵ -dense discrete subset $Z \subset X$. For given L there is an open cover \mathcal{U} of Z with the Lebesgue number L of multiplicity $d(L)$. Let $\mathcal{U} = \{U_i\}_{i \in J}$. We define $\phi_i(x) = \frac{d(x, Z \setminus U_i)}{\sum_{i \in J} d(x, Z \setminus U_i)}$. Denote by $\Sigma_x = \sum_{i \in J} d(x, Z \setminus U_i)$. For every i we fix $y_i \in U_i$. Define $a^L(z) = \sum_{i \in J} \phi_i(z) \delta_{y_i} \in P(Z)$. Note that $\text{supp}(a^L(z)) = \{y_i \mid \phi_i(z) \neq 0\} = \{y_i \mid z \in U_i\} \subset \{z' \in Z \mid d(z, z') < R\}$ where $\text{mesh} \mathcal{U} < R$.

To check (2) note that

$$\|a^L(z) - a^L(w)\|_1 = \|\sum_{i \in J} \phi_i(z) \delta_{y_i} - \sum_{i \in J} \phi_i(w) \delta_{y_i}\|_1 = \|\sum_{i \in J} (\phi_i(z) - \phi_i(w)) \delta_{y_i}\|_1$$

$$\leq \sum_J |\phi_i(z) - \phi_i(w)| = \sum_J \left| \frac{d(z, Z \setminus U_i)}{\Sigma_z} - \frac{d(w, Z \setminus U_i)}{\Sigma_w} \right| \leq \sum_J \left| \frac{d(z, Z \setminus U_i)}{\Sigma_z} - \frac{d(w, Z \setminus U_i)}{\Sigma_z} \right| +$$

$$\sum_J d(w, Z \setminus U_i) \left| \frac{1}{\Sigma_z} - \frac{1}{\Sigma_w} \right| \leq \frac{1}{\Sigma_z} \sum_J |d(z, Z \setminus U_i) - d(w, Z \setminus U_i)| + \sum_w \left| \frac{1}{\Sigma_z} - \frac{1}{\Sigma_w} \right| \leq$$

$$\frac{2d(L)}{\Sigma_z} d(z, w) + \frac{|\Sigma_w - \Sigma_z|}{\Sigma_z} \leq \frac{2d(L)}{L} d(z, w) + \frac{2d(L)}{L} d(z, w) \leq 4K \frac{d(L)}{L} \rightarrow 0 \text{ as } L \rightarrow \infty. \quad \square$$

In classical dimension theory of infinite dimensional spaces there is a special class of spaces having the Property C. Compact metric spaces with the Property C are having some features of finite dimensional spaces (see [vM-M]). Below we define an asymptotical analog of the Property C.

Definition. *A metric space $X \in BG$ has an asymptotic property C if for every sequence of numbers $R_1 \leq R_2 \leq R_3 \dots$ there exists a finite sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=1}^k$ of open subsets of X such that the union $\cup_{i=1}^k \mathcal{U}_i$ is a cover of X and every family \mathcal{U}_i is R_i -disjoint.*

The following theorem is close to Theorem 8.9.

Theorem 8.11. *The Coarse Baum-Connes conjecture holds for metric spaces with the asymptotic property C.*

Proof. We apply Lemma 8.10 and the Theorem of Yu. For every n we define a map $a^n : X \rightarrow P(X)$ as follows. First we assume that X is a discrete space. Let $R_i = n^i$ and let $\mathcal{U}_1, \dots, \mathcal{U}_k$ be a sequence of $2R_i$ -disjoint uniformly bounded families from the definition of the asymptotic property C. For every $U_j^i \in \mathcal{U}_i$ we fix a point $x_j^i \in U_j^i$. We define

$$\phi_j^i(x) = \begin{cases} \frac{R_i}{2}, & \text{if } x \in U_j^i; \\ \max\{0, \frac{R_i}{2} - d(x, U_j^i)\}, & \text{otherwise} \end{cases}$$

and define $b^n(x) = \sum_{i,j} n^{k-i+1} \phi_j^i(x) \delta_{x_j^i}$. We set $a^n(x) = \frac{b^n(x)}{\|b^n(x)\|_1}$. Note that for every $x \in X$ and any i there is at most one $j = j_x(i)$ such that $\phi_j^i(x) \neq 0$. Then $\text{supp}(a^n(x))$

is contained in the ball $B_r(x)$ with $r = \max\{\text{diam}\mathcal{U}_i + 2R_i\}$. Thus the condition (1) of Lemma 8.10 holds.

In order to verify the condition (2) first we show that $\|b_x^n\|_1 \geq \frac{n^{k+1}}{2}$ for all x . Indeed, $\|b_x^n\|_1 = \sum_i n^{k-i+1} |\phi_{j(i)}^i(x)| \geq n^{k-i+1} \frac{R_i}{2} = \frac{n^{k+1}}{2}$ where i is chosen with $x \in U_{j(i)}^i$.

Without loss of generality we may assume that $\|b_y^n\|_1 \geq \|b_x^n\|_1$. Then $\|b_x^n - \|b_x^n\|_1 a_y^n\|_1 \leq \|\frac{\|b_x^n\|_1}{\|b_y^n\|_1} b_y^n - b_x^n\|_1 + \|b_y^n - b_x^n\|_1$

$$= \|b_y^n\|_1 \frac{|\|b_x^n\|_1 - \|b_y^n\|_1|}{\|b_y^n\|_1} + \|b_y^n - b_x^n\|_1 \leq 2\|b_y^n - b_x^n\|_1$$

$$2\|\sum_i n^{k-i+1} \phi_{j_x(i)}^i(x) \delta_{x_{j_x(i)}^i} - \sum_i n^{k-i+1} \phi_{j_y(i)}^i(y) \delta_{x_{j_y(i)}^i}\|_1 \leq$$

$$2\|\sum_{i, j(i)=j_x(i)=j_y(i)} n^{k-i+1} (\phi_{j(i)}^i(x) - \phi_{j(i)}^i(y)) \delta_{x_{j(i)}^i}\|_1 +$$

(here we consider $j(i)$ equal to any of the indexes $j_x(i)$ or $j_y(i)$ when one of them is not defined)

$$2\|\sum_{i, j_x(i) \neq j_y(i)} n^{k-i+1} \phi_{j_x(i)}^i(x) \delta_{x_{j_x(i)}^i}\|_1 + 2\|\sum_{i, j_x(i) \neq j_y(i)} n^{k-i+1} \phi_{j_y(i)}^i(y) \delta_{x_{j_y(i)}^i}\|_1$$

$$\leq 2\sum_i n^{k-i+1} |\phi_{j(i)}^i - \phi_{j(i)}^i(y)| + 4\sum_i n^{k-i+1} d(x, y).$$

Here we used that for $j_x(i) \neq j_y(i)$ we have $d(x, y) \geq R_i \geq 2\phi_j^i$.

Note that $|\phi_{j(i)}^i - \phi_{j(i)}^i(y)| \leq d(x, y)$. Then

$$\sum_i n^{k-i+1} |\phi_{j(i)}^i - \phi_{j(i)}^i(y)| \leq \sum_{i=1}^k n^{k-i+1} d(x, y) = n \frac{n^k-1}{n-1} d(x, y).$$

Then $\|b_x^n - \|b_x^n\|_1 a_y^n\|_1 \leq 6n \frac{n^k-1}{n-1} d(x, y)$. Therefore $\|a_x^n - a_y^n\|_1 = \frac{1}{\|b_x^n\|_1} \|b_x^n - \|b_x^n\|_1 a_y^n\|_1 \leq$

$$\frac{6n}{\|b_x^n\|_1} \frac{n^k-1}{n-1} d(x, y) \leq \frac{12n}{n^{k+1}} \frac{n^k-1}{n-1} d(x, y) \leq \frac{12}{n-1} d(x, y).$$

If $d(x, y) \leq K$ we have that $\lim_{n \rightarrow \infty} \frac{12n}{n^{k+1}} \frac{n^k-1}{n-1} d(x, y) = 0$. \square

§9 PRINCIPLE OF DESCENT AND THE HIGSON CORONA

The Descent Principle is the statement that the original Novikov higher signature conjecture for geometrically finite group can be derived from its coarse counterpart. The main coarse analog of the Novikov Conjecture is the coarse Baum-Connes conjecture which was considered in the previous section. Here we consider coarser statements which also imply the Novikov Conjecture. Let X be a universal cover of finite aspherical polyhedron $B\Gamma$ supplied with a metric lifted from $B\Gamma$. Then each of the following four conditions implies the Novikov conjecture for Γ .

(CPI) [C-P]. *There is an equivariant rationally acyclic metrizable compactification \hat{X} of X such that the action of Γ is small at infinity.*

(CPII) [C-P2]. *There is an equivariant rationally acyclic (possibly nonmetrizable) compactification \hat{X} of X with a system of covers α of $Y = \hat{X} \setminus X$ by boundedly saturated sets*

such that the projection to the inverse limits of the nerves of α induces an isomorphism $H_*(Y; \mathbf{Q}) \rightarrow H_*(\lim_{\leftarrow} N(\alpha); \mathbf{Q})$.

(FW) [F-W],[D-F]. *The boundary homomorphism $\delta : H^{n-1}(\nu X; \mathbf{Q}) \rightarrow H_c^n(X; \mathbf{Q})$ is an equivariant split surjection.*

(HR) [Ro1]. *There is an equivariant rationally acyclic Higson dominated compactification \hat{X} of X .*

An action of Γ is *small at infinity* for a given compactification \bar{X} of X if for every $x \in \bar{X} \setminus X$ and a neighborhood U of x in \bar{X} , for every compact set $C \subset X$ there is a smaller neighborhood V such that $g(C) \cap V \neq \emptyset$ implies $g(C) \subset U$ for all $g \in \Gamma$.

An open set $U \subset Y = \hat{X} \setminus X$ is called *boundedly saturated* if for every closed set $C \subset \hat{X}$ with $C \cap Y \subset U$ the closure of any r -neighborhood $N_r(C \cap X)$ satisfies $\overline{N_r(C \cap X)} \cap Y \subset U$.

We consider a Γ -invariant metric on X . Since $B\Gamma$ is a finite complex, the Higson corona of X does not depend on choice of metric and coincides with the Higson corona of Γ .

Proposition 9.1 [D-F]. *The action of Γ on X is small at infinity for a compactification \bar{X} if and only if \bar{X} is Higson dominated.*

Let Γ be a finitely generated group with the word metric $d(a, b) = l(a^{-1}b)$ where $l(w)$ denotes the minimal length of a word presenting $w \in \Gamma$ with a fixed finite set of generators. A group Γ acts on itself by isometries: $g : \Gamma \rightarrow \Gamma$ by the formula $g(a) = ga$. This action is called the *left action*.

Proposition 9.2. *The left action of Γ on itself can be extended to an action on the Higson compactification $\bar{\Gamma}$.*

Proof. An isometry $g : \Gamma \rightarrow \Gamma$ is extendible over the Higson corona to a continuous map $g : \bar{\Gamma} \rightarrow \bar{\Gamma}$. This implies that the whole action of Γ is extendible.

Lemma 9.3. *Let X be as above. The following are equivalent.*

- (1) *The boundary homomorphism $\delta : H^{n-1}(\nu X; \mathbf{Q}) \rightarrow H_c^n(X; \mathbf{Q})$ is an equivariant split surjection.*
- (2) *There is a Higson dominated equivariant compactification \hat{X} of X such that the boundary homomorphism $\hat{\delta} : H^{n-1}(\hat{X} \setminus X; \mathbf{Q}) \rightarrow H_c^n(X; \mathbf{Q})$ is an equivariant split surjection.*
- (3) *There is a metrizable Higson dominated equivariant compactification \hat{X} of X such that the boundary homomorphism $\hat{\delta} : H^{n-1}(\hat{X} \setminus X; \mathbf{Q}) \rightarrow H_c^n(X; \mathbf{Q})$ is an equivariant split surjection.*
- (4) *There is an equivariant metrizable Higson dominated compactification \hat{X} of X such that the boundary homomorphism $H_*^{lf}(X; \mathbf{Q}) \rightarrow H_{*-1}^s(\hat{X} \setminus X; \mathbf{Q})$ is an equivariant split injection.*

Here H_*^s stands for the Steenrod homology.

Proof. 1) \Rightarrow 2). Because of Proposition 9.2 this implication follows.

2) \Rightarrow 1). Let $\xi^* : H^{n-1}(\hat{X} \setminus X; \mathbf{Q}) \rightarrow H^{n-1}(\nu X; \mathbf{Q})$ be a homomorphism generated by the domination $\xi : \bar{X} \rightarrow \hat{X}$. The map ξ is equivariant, since it is equivariant on a dense subset X . Let $s' : H_c^n(X; \mathbf{Q}) \rightarrow H^{n-1}(\hat{X} \setminus X; \mathbf{Q})$ be an equivariant splitting of the boundary homomorphism. Then $\xi^* \circ s'$ is an equivariant splitting of $\delta : H^{n-1}(\nu X; \mathbf{Q}) \rightarrow H_c^n(X; \mathbf{Q})$.

3) \Rightarrow 2). Obvious.

3) \Leftrightarrow 4). This is the standard duality (of vector spaces) between homology and cohomology with coefficients in a field.

For a proof of the implication 2) \Rightarrow 3) we need to develop some technique.

A *directed* set A is a partially ordered set with the property that for any two elements $\alpha, \beta \in A$ there exists $\gamma \in A$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$. A directed set is called σ -*complete* if for any countable chain $C \subset A$ there is a supremum $\sup C \in A$. A subset $B \subset A$ of a σ -complete set A is called σ -*closed* if for any countable chain C in B we have $\sup C \in B$. A subset $B \subset A$ is called *cofinal* if for any $\alpha \in A$ there is $\beta \in B$ with $\beta \geq \alpha$.

Proposition 9.4. *The intersection of countably many σ -closed cofinal subsets $\cap B_i \subset A$ is σ -closed cofinal.*

Proof. It is clear that $\cap B_i$ is σ -closed. First we show that the intersection of two σ -closed cofinal sets B and B' is cofinal. For that for any $\alpha \in A$ we construct a sequence β_i such that 1) $\beta_0 = \alpha$, $\beta_{2k-1} \in B$ and $\beta_{2k} \in B'$, $k > 0$; 2) $\beta_i < \beta_{i+1}$. Then $\sup\{\beta_i\} \in B \cap B'$.

Next, we construct a sequence α_i such that: 1) $\alpha_i \leq \alpha_{i+1}$, 2) $\alpha_i \in \cap_k^i B_k$ and $\alpha_0 \in A$ is given. Then $\sup\{\alpha_i\} \in \cap B_i$. \square

An inverse (direct) system in a given category \mathcal{C} over a σ -complete ordered set A is called σ -*continuous* if

$$\lim_{\leftarrow} \{X_{\alpha_i} \mid \alpha_i \in C\} = X_{\sup C} \quad (\lim_{\rightarrow} \{H_{\alpha_i} \mid \alpha_i \in C\} = X_{\sup C})$$

for every countable chain C .

Schepin Spectral Theorem. *[Sc],[Ch]. Let $\{X_\alpha \mid \alpha \in A\}$ and $\{Y_\alpha \mid \alpha \in A\}$ be inverse σ -continuous systems of compact metric spaces.*

1) *Existence.* Let $f : X = \lim\{X_\alpha\} \rightarrow Y = \lim\{Y_\alpha\}$ be a continuous map. Then there exists a σ -closed cofinal subset $B \subset A$ and a morphism of spectra

$$\{f_\beta\}_{\beta \in B} : \{X_\beta \mid \beta \in B\} \rightarrow \{Y_\beta \mid \beta \in B\} \text{ such that } f = \lim_{\leftarrow} \{f_\beta \mid \beta \in B\}.$$

2) *Uniqueness.* Let $\{f_\alpha\}_{\alpha \in A}$ and $\{g_\alpha\}_{\alpha \in A}$ be two morphisms between the spectra $\{X_\alpha \mid \alpha \in A\}$ and $\{Y_\alpha \mid \alpha \in A\}$ with $\lim_{\leftarrow} \{f_\alpha\} = \lim_{\leftarrow} \{g_\alpha\}$. Then there exists a σ -closed cofinal subset $B \subset A$ such that $f_\beta = g_\beta$ for $\beta \in B$.

Corollary. *For any homeomorphism $h : X = \lim\{X_\alpha\} \rightarrow Y = \lim\{Y_\alpha\}$ there exists a σ -closed cofinal subset $B \subset A$ and an isomorphism of spectra $\{h_\beta\}_{\beta \in B} : \{X_\beta \mid \beta \in B\} \rightarrow \{Y_\beta \mid \beta \in B\}$ such that $h = \lim_{\leftarrow} \{h_\beta \mid \beta \in B\}$.*

Proof. Apply existence part for h and h^{-1} and then apply Proposition 9.4.

For every compact Hausdorff space X one can define a continuous σ -spectrum $\{X_\alpha \mid \alpha \in A\}$ as follows. Consider an imbedding $X \subset I^B$. Let A be the set of all countable subsets of B , define $X_\alpha = \pi_\alpha(X)$, where $\pi_\alpha : I^B \rightarrow I^\alpha$ is a projection, $\alpha \in A$. All bonding maps are defined similarly.

There is a dual theorem to the Schepin Spectral theorem for σ -continuous direct system of countable CW-complexes [Dr-Dy1]. In this paper we need the dual theorem in the category of abelian groups.

Dual Spectral Theorem. *Let $\{H_\alpha \mid \alpha \in A\}$ and $\{G_\alpha \mid \alpha \in A\}$ be direct σ -continuous systems of countable abelian groups.*

1) *Existence.* Let $f : H = \lim\{H_\alpha\} \rightarrow G = \lim\{G_\alpha\}$ be a homomorphism. Then there exists a σ -closed cofinal subset $B \subset A$ and a morphism of spectra

$$\{f_\beta\}_{\beta \in B} : \{H_\beta \mid \beta \in B\} \rightarrow \{G_\beta \mid \beta \in B\} \text{ such that } f = \lim_{\rightarrow} \{f_\beta \mid \beta \in B\}.$$

2) *Uniqueness.* Let $\{f_\alpha\}_{\alpha \in A}$ and $\{g_\alpha\}_{\alpha \in A}$ be two morphisms between the spectra $\{H_\alpha \mid \alpha \in A\}$ and $\{G_\alpha \mid \alpha \in A\}$ with $\lim_{\rightarrow} \{f_\alpha\} = \lim_{\rightarrow} \{g_\alpha\}$. Then there exists a σ -closed cofinal subset $B \subset A$ such that $f_\beta = g_\beta$ for $\beta \in B$.

Proof. Let $p_\alpha^\beta : H_\alpha \rightarrow H_\beta$ and $q_\alpha^\beta : G_\alpha \rightarrow G_\beta$ be the bonding maps, $\alpha \leq \beta$. Denote $p_\alpha = \lim_{\beta} p_\alpha^\beta : H_\alpha \rightarrow H$ and $q_\alpha = \lim_{\beta} q_\alpha^\beta : G_\alpha \rightarrow G$

1). First we show that the set $M_H = \{\alpha \in A \mid \text{Ker}(p_\alpha) = 0\}$ is cofinal in A . For any $\alpha_0 \in A$ we built the chain $\alpha_0 < \alpha_1 < \dots$ such that $\text{Ker}(p_{\alpha_{i+1}}) = \text{Ker}(p_{\alpha_i})$. Let $\alpha = \sup\{\alpha_i\}$ and let $a \in \text{Ker}(p_\alpha)$. Since the spectrum is σ -continuous, $a = p_{\alpha_i}^\alpha(c)$ for some i and some c . Then $c \in \text{Ker}(p_{\alpha_i})$, therefore $p_{\alpha_i}^{\alpha_{i+1}}(c) = 0$. Hence $a = 0$.

Let $M = M_H \cap M_G$. Note that M is σ -closed and cofinal. We show that the set $B = \{\beta \in M \mid \text{there exists a homomorphism } f_\beta : H_\beta \rightarrow G_\beta \text{ with } f \circ p_\beta = q_\beta \circ f_\beta\}$ is cofinal in A . Let $\alpha \in A$ be given. For any $a \in H_\alpha$ there is $\beta(a) \in A$ such that $f p_\alpha(a) = q_{\beta(a)}(b(a))$ for some element $b(a) \in G_{\beta(a)}$. Enumerate elements of H_α as a_1, a_2, \dots and construct a sequence $\alpha < \beta(a_1) < \beta(a_2) < \dots$. Let $\gamma = \sup\{\beta(a_i)\}$. Then we define a map $f_{\alpha\gamma} : H_\alpha \rightarrow G_\gamma$ by the formula $f_{\alpha\gamma}(a) = q_{\beta(a)}^\gamma(b(a))$. Note that $f_{\alpha\gamma}$ is a homomorphism with $f \circ p_\alpha = q_\gamma \circ f_{\alpha\gamma}$. The latter is obvious, we check that $f_{\alpha\gamma}$ is a homomorphism. Since q_γ is a monomorphism, the equalities $q_\gamma f_{\alpha\gamma}(a+a') = f p_\alpha(a+a') = f p_\alpha(a) + f p_\alpha(a') = q_\gamma f_{\alpha\gamma}(a) + q_\gamma f_{\alpha\gamma}(a')$ imply that $f_{\alpha\gamma}(a+a') = f_{\alpha\gamma}(a) + f_{\alpha\gamma}(a')$. Define $\alpha_1 = \gamma$ and repeat the procedure to obtain α_2 and so on. We will obtain a chain $\alpha < \alpha_1 < \alpha_2 < \dots$ and a sequence of homomorphisms $f_{\alpha_i \alpha_{i+1}} : H_{\alpha_i} \rightarrow G_{\alpha_{i+1}}$ such that all squares are commutative. Then there exists a homomorphism $f_{\bar{\alpha}} : H_{\bar{\alpha}} \rightarrow G_{\bar{\alpha}}$ with $f_{\bar{\alpha}} = \lim_{\rightarrow} f_{\alpha_i \alpha_{i+1}}$ such that $f p_{\bar{\alpha}} = q_{\bar{\alpha}} f_{\bar{\alpha}}$. Clearly B is σ -closed and $\{f_\beta\}_{\beta \in B}$ is a morphism of spectra.

2). Take $B = M$, then the result follows.

Proof of 2) \Rightarrow 3) of Lemma 9.3. According to the remark after Schepin Spectral Theorem we can present \hat{X} as the inverse limit of a σ -continuous system of compact metric spaces $\hat{X} = \lim_{\leftarrow} \{\hat{X}_\alpha; p_\alpha^\beta \mid \alpha \in A'\}$. It is easy to verify that the set $A = \{\alpha \mid p_\alpha^{-1} p_\alpha(X) = X\}$ is σ -closed cofinal in A' . For any element $\gamma \in \Gamma$ we apply the Schepin Spectral Theorem to a homeomorphism $\gamma : \hat{X} \rightarrow \hat{X}$ to obtain a σ -closed cofinal subset $B_\gamma \subset A$ with

corresponding isomorphism of spectra. By Proposition 9.4 the set $B = \bigcap_{\gamma \in \Gamma} B_\gamma$ is σ -closed cofinal. Applying the uniqueness part of the Schepin Spectral Theorem, we may assume that \hat{X}_α is an equivariant compactification of X such that all bonding maps are equivariant.

The homomorphism $\hat{\delta}$ is induced by the quotient map $\hat{X} \cup \text{cone}(\hat{X} \setminus X) \rightarrow \Sigma(\hat{X} \setminus X)$. So, $\hat{\delta} : G = H^n(\Sigma(\hat{X} \setminus X); \mathbf{Q}) \rightarrow H = H^n(\hat{X} \cup \text{cone}(\hat{X} \setminus X); \mathbf{Q})$. The condition 2) says that $\hat{\delta}$ admits an equivariant splitting $s : H \rightarrow G$ i.e. $\hat{\delta} \circ s = \text{id}$. By part one of the Dual Spectral Theorem there exists $B_1 \subset B$ such that s is the limit of a morphism $\{s_\beta : H_\beta \rightarrow G_\beta \mid \beta \in B_1\}$. By part two of the Dual Spectral Theorem we may assume that $\gamma^{-1}s_\beta\gamma = s_\beta$ and $\hat{\delta}_\beta s_\beta = \text{id}$ for all $\beta \in B_1$ and all $\gamma \in \Gamma$. Then for any $\beta \in B_1$ a compact \hat{X}_β can serve as a metrizable Higson dominated compactification required in 3).

Let c_1X and c_2X be two compactifications of X , the maximal compactification cX dominated by both c_1X and c_2X is called the minimum of c_1X and c_2X and is denoted as $\min\{c_1X, c_2X\}$. We recall that compactifications of X are in one-to-one correspondence with totally bounded uniformities on X [En]. Two sets $A, B \subset X$ have nonempty intersection at infinity, $\bar{A} \cap \bar{B} \neq \emptyset$, if and only if $(A \times B) \cap V \neq \emptyset$ for all entourages $V \in \mathcal{U}$. Let \mathcal{U}_1 and \mathcal{U}_2 be two uniformities on X corresponding compactifications c_1X and c_2X , then the uniformity \mathcal{U} corresponding to $\min\{c_1X, c_2X\}$ is defined as $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$.

For a compactification cX of locally compact space X we denote by $\mathcal{P}(cX)$ the set of pairs (A, B) of closed sets in X with $\bar{A} \cap \bar{B} \neq \emptyset$.

Proposition 9.5. *Let X be a locally compact space, let c_1X and c_2X be two compactifications of X and let $cX = \min\{c_1X, c_2X\}$. Then $\mathcal{P}(cX) = \mathcal{P}(c_1X) \cup \mathcal{P}(c_2X)$.*

Proof. It is clear that $\mathcal{P}(cX) \supset \mathcal{P}(c_1X) \cup \mathcal{P}(c_2X)$. Let $(A, B) \in \mathcal{P}(cX)$. Assume that $(A, B) \notin \mathcal{P}(c_iX)$, $i = 1, 2$. Then there exist $V_i \in \mathcal{U}_i$ such that $(A \times B) \cap V_i = \emptyset$. Therefore $(A \times B) \cap (V_1 \cup V_2) = \emptyset$. By the definition of uniformity $V_1 \cup V_2 \in \mathcal{U}_i$ for $i = 1, 2$. Then $V_1 \cup V_2 \in \mathcal{U}$. Contradiction. \square

The homomorphism $H_*(Y; \mathbf{Q}) \rightarrow H_*(\lim_{\leftarrow} N(\alpha); \mathbf{Q})$ in *CPII* is an isomorphism if the system $\{\alpha\}$ is cofinal. We introduce the condition.

(CPII'). There is an equivariant rationally acyclic (possibly nonmetrizable) compactification \hat{X} of X with a cofinal system of covers α of $Y = \hat{X} \setminus X$ by boundedly saturated sets.

We denote by *CPI'* the condition *CPI* without an assumption of metrizability of \hat{X} .

Theorem 9.6. $CPII' \Leftrightarrow CPI' \Leftrightarrow CPI \Leftrightarrow HR \Rightarrow FW \Leftarrow CPII$.

Proof. By Proposition 9.1 the property of a compactification to be Higson dominated is equivalent to the small action at infinity condition. Hence $HR \Leftrightarrow CPI'$. The exact sequence of pair $(\hat{X}, \hat{X} \setminus X)$ implies that $HR \Rightarrow FW$. Clearly, $CPI \Rightarrow CPI'$. Show that $CPI \Leftarrow CPI'$. Let \hat{X} be a rationally acyclic compactification of X . We can present \hat{X} as

the limit space of an inverse σ -continuous system of metrizable compacta $\{\hat{X}_\alpha \mid \alpha \in A\}$. Like in the proof of $2) \Rightarrow 3)$ of Lemma 9.3 we may assume that each \hat{X}_α is an equivariant compactification of X . Consider the direct system $\{H^i(\hat{X}_\alpha; \mathbf{Q}) \mid \alpha \in A\}$. Let $M_{H^i} \subset A$ be the set defined in the proof of the Dual Spectral Theorem. We recall that by the definition for any $\alpha \in M_{H^i}$, the kernel of $p_\alpha^* : H^i(\hat{X}_\alpha; \mathbf{Q}) \rightarrow H^i(\hat{X}; \mathbf{Q})$ is trivial. Since $H^i(\hat{X}; \mathbf{Q}) = 0$, it follows that every \hat{X}_α is rationally acyclic.

Next we show that $CPII \Rightarrow FW$. Let cX be $\min\{\hat{X}, \bar{X} = \nu X \cup X\}$ and let $g : \hat{X} \rightarrow cX$ be the natural projection. First we show that every boundedly saturated open set $U \subset Y$ is of the form $g^{-1}(V)$. This implies that V is open. Show that $g(U) = g(Y) \setminus g(Y \setminus U)$. Let $x \in g(U)$ and $x = g(z)$, $z \in U$. Let F be a closed set in \hat{X} such that $F \cap (Y \setminus U) = \emptyset$ and $z \in F$. Let H be a closed set in \hat{X} such that $H \cap Y = Y \setminus U$. Note that if $g(F) \cap g(Y \setminus U) = \emptyset$, then $x \in g(Y) \setminus g(Y \setminus U)$. Assume the contrary $g(F) \cap g(Y \setminus U) \neq \emptyset$. Then $g(F) \cap g(H) \neq \emptyset$ and hence $(g(F) \cap X, g(H) \cap X) = g(F \cap X, H \cap X) \in \mathcal{P}(cX)$. Since $F \cap H = \emptyset$, we have $(F \cap X, H \cap X) \notin \mathcal{P}(\hat{X})$. Proposition 9.5 implies that $(F \cap X, H \cap X) \in \mathcal{P}(\bar{X})$. Then there exist a constant c and sequences $\{x_n\} \subset F \cap X$ and $\{y_n\} \subset H \cap X$, tending to infinity, and such that $\overline{dist(x_n, y_n)} < c$. Since U is boundedly saturated, we have $\overline{N_{2c}(F \cap X)} \cap Y \subset U$. Hence $\overline{N_{2c}(F \cap X)} \cap (Y \setminus U) = \emptyset$. Hence $(N_{2c}(F \cap X) \cap H) \setminus B_R(x_0) = \emptyset$ for large enough R . This contradicts to the fact that $\{y_n\}$ tends to infinity.

Therefore for any covering α of Y by boundedly saturated subsets the image $g(\alpha)$ is an open cover of $cX \setminus X$ with the same nerve. Consider the following diagram:

$$\begin{array}{ccc}
H_{k+1}^{lf}(X; \mathbf{Q}) & \xlongequal{\quad} & H_{k+1}^{lf}(X; \mathbf{Q}) \\
\downarrow = & & \downarrow \partial \\
H_k(Y; \mathbf{Q}) & & H_k(cX \setminus X; \mathbf{Q}) \\
\downarrow = & & \downarrow \\
H_k(\lim_{\leftarrow} N(\alpha); \mathbf{Q}) & \xrightarrow{\quad} & H_k(\lim_{\leftarrow} N(g(\alpha)); \mathbf{Q})
\end{array}$$

The homomorphism ∂ is an equivariant split injection as a left divisor of an equivariant isomorphism. For cohomology it is an equivariant split surjection and hence FW holds.

The condition $CPII'$ implies that the corresponding compactification is Higson dominated. \square

Thus the condition FW is the weakest among above. The condition FW is a further coarsening of the following version of the Coarse Baum-Connes conjecture: *the rational Roe index map $K_*^{lf}(X) \otimes \mathbf{Q} \rightarrow K_*(C(X)) \otimes \mathbf{Q}$ is an equivariant split monomorphism* which also implies the Novikov conjecture.

§10 OPEN PROBLEMS

Dimension and Higson corona. The connection between asymptotic dimensions of metric space and dimensions of its (nonmetrizable) Higson corona is not fully investi-

gated yet. The problems in this paragraph are of particular interest for metric spaces of bounded geometry.

Problem 1. *Is it always true that $as\dim X = \dim \nu X$?*

In view of Theorem 7.2 and [D-K-U] this problem can be reformulated as follows:
Does there exist a metric space X of infinite asymptotic dimension with $\dim \nu X < \infty$?

Problem 2. *Does there exist a metric space X of an infinite asymptotic dimension with $as\dim_* X < \infty$?*

A positive answer to Problem 2 gives a negative answer to Problem 1. A relevant question is whether $as\dim_* X$ and $\dim \nu X$ always agree.

Problem 3. *Does the inequality $\dim_G \nu X \leq as\dim_G X$ hold for all metric spaces X and all abelian groups G ?*

Problem 4. *Is it true that $\dim_{\mathbf{Z}} \nu \Gamma < \infty$ for all geometrically finite groups Γ ?*

An affirmative answer to Problem 3 would imply an affirmative answer to Problem 4. Using analogy one can define an asymptotic inductive dimension $asInd$.

Problem 5. *What is the relation between $asInd X$ and other dimensions: $as\dim X$, $as\dim_* X$ and $\dim \nu X$?*

Problem 6. *Let X be a metric space of bounded geometry with slow dimension growth and with $asdim X = \infty$. Does it follow that $\dim \nu X = \infty$?*

Problem 7. *Can a metric space X with $as\dim X = n$ be coarsely uniformly embedded in a simply connected $(2n + 1)$ -dimensional non-positively curved manifold?*

The answer is 'yes' for $2n + 2$ -dimensional manifolds.

Large scale Alexandroff Problem.

Problem 8. *Does the equality $as\dim_{\mathbf{Z}} \Gamma = as\dim \Gamma$ hold for geometrically finite groups Γ ?*

The spectrum $\mathbb{S} = \{\Omega^\infty \Sigma^\infty S^n\}$ defines stable cohomotopy. Then the stable cohomotopy dimension $\dim_{\mathbb{S}} X$ can be defined in terms of extensions of maps to $\Omega^\infty \Sigma^\infty S^n$. Namely, $\dim_{\mathbb{S}} X \leq n$ if and only if every continuous map $f : A \rightarrow \Omega^\infty \Sigma^\infty S^n$ defined on a closed subset $A \subset X$ can be extended over all X . It is easy to check that for every finite m extensions to $\Omega^m \Sigma^m S^n$ classify the covering dimension \dim . Thus, $\dim X$ and $\dim_{\mathbb{S}} X$ are very close. One can define a macroscopic version of the stable cohomological dimension as an asymptotic generalized cohomological dimension.

Problem 9. *Does the asymptotic stable cohomotopy dimension $as\dim_{\mathbb{S}} \Gamma$ coincide with $as\dim \Gamma$ for geometrically finite groups?*

Positive answers to Problems 8,9 imply the Novikov Conjecture.

Problem 10. *Does the equality $\dim_{\mathbf{Z}} X = \dim X$ hold for compact H -spaces ?*

Problem 11. *Does the equality $\dim_{\mathbf{Z}} G = \dim G$ hold for topological (noncompact) groups G ?*

Problem 12. *What it would be a coarse analogue of the Edwards resolution theorem?*

Absolute neighborhood extensors.

Problem 13. *Is the space of probability measures $P(X)$ an absolute extensor in the asymptotic topology?*

It is not difficult to show that $P(X)$ is AE in the class of finite dimensional spaces or in the class of spaces with the slow dimension growth.

Problem 14. *Does the Homotopy Extension Theorem hold in the asymptotic category in full generality?*

An affirmative answer to Problem 13 implies an affirmative answer to Problem 14.

Problem 15. *Prove a macroscopic analog of the West theorem stating that every ANE is homotopy equivalent to a polyhedron.*

Fragments of the Micro-Macro topology dictionary.

MICRO	MACRO
1. Compactum i.e. compact metrizable space	Proper metric space of bounded geometry
2. Alexandroff-Čech approximation by polyhedra	Anti-Čech approximation by polyhedra
3. Lebesgue dimension \dim	Gromov dimension <i>as</i> \dim
4. Alexandroff characterization of \dim by maps to S^n	\dim^c =covering dimension of the Higson corona
5. Local contractibility	Uniform contractibility
6. Local n -connectivity	Uniform n -connectivity
7. Neighborhood	Asymptotic neighborhood

8. ANE	ANE
9. Polyhedron	Asymptotic polyhedron or open cone over a polyhedron
10. One point space	\mathbf{R}_+ (as well as \mathbf{N})
11. Unit interval $[0, 1]$	\mathbf{R}_+^2
12. n -Sphere S^n	\mathbf{R}^{n+1}
13. ?	Hyperbolic space \mathbf{H}^n
14. Homotopy	Homotopy in \mathcal{A}
15. Cohomology	anti-Čech cohomology Roe's cohomology
16. Čech homology Steenrod homology	Coarse homology
17. Cohomological dimension	Asymptotic cohomological dimension
18. Fundamental group	Asymptotic fundamental group can be defined by using \mathbf{R}_+^2 instead of $[0, 1]$
19. Manifold	Open contractible manifold?
20. ?	Discrete group

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